

Cosmological perturbations on the Phantom brane

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We obtain a closed system of equations for scalar perturbations in a multi-component braneworld. Our braneworld possesses a phantom-like equation of state at late times, $w_{\text{eff}} < -1$, but no big-rip future singularity. In addition to matter and radiation, the braneworld possesses a new effective degree of freedom – the ‘Weyl fluid’ or ‘dark radiation’. Setting initial conditions on super-Hubble spatial scales at the epoch of radiation domination, we evolve perturbations of radiation, pressureless matter and the Weyl fluid until the present epoch. We observe a gradual decrease in the amplitude of the Weyl-fluid perturbations after Hubble-radius crossing, which results in a negligible effect of the Weyl fluid on the evolution of matter perturbations on spatial scales relevant for structure formation. Consequently, the quasi-static approximation of Koyama and Maartens provides a good fit to the exact results during the matter-dominated epoch. We find that the late-time growth of density perturbations on the brane proceeds at a faster rate than in Λ CDM. Additionally, the gravitational potentials Φ and Ψ evolve differently on the brane than in Λ CDM, for which $\Phi = \Psi$. On the brane, by contrast, the ratio Φ/Ψ exceeds unity during the late matter-dominated epoch ($z \lesssim 50$). These features emerge as *smoking gun* tests of phantom brane cosmology and allow predictions of this scenario to be tested against observations of galaxy clustering and large-scale structure.

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I. INTRODUCTION

Cosmology, during the past two decades, has witnessed the introduction and development of several bold new theoretical ideas. One, especially radical paradigm, involves the braneworld concept. According to this paradigm (see [1] for a review), our universe is a

lower-dimensional hypersurface (the ‘brane’) embedded in a higher-dimensional spacetime (the ‘bulk’). A new feature of the braneworld paradigm, which distinguishes it from the earlier Kaluza–Klein constructs, is that spacetime dimensions orthogonal to the brane need not be compact but could be ‘large’ [2] and even infinite in length. In the simplest and most thoroughly investigated cosmological models, there is only one large extra dimension accessible to gravity, while all standard-model fields are assumed to be trapped on the brane. From the viewpoint of our four-dimensional world, this manifests as a modification of gravity. In the seminal Randall–Sundrum (RS) model [3], gravity is modified on relatively small spatial scales. Apart from other interesting applications, this model was used to provide an alternative explanation of galactic rotation curves and X-ray profiles of galactic clusters without invoking the notion of dark matter [4].

An important class of braneworld models contains the so-called ‘induced-gravity’ term in the action for the brane (it is induced by quantum corrections from the matter fields, hence the term), and modifies gravity on relatively large spatial scales. First proposed in [5–7], it has become known as the Dvali–Gabadadze–Porrati (DGP) model. Depending on the embedding of the brane in the bulk space, this model has two branches of cosmological solutions [8]. The ‘self-accelerating’ branch was proposed to describe cosmology with late-time acceleration without bulk and brane cosmological constants [9], while the ‘normal’ branch requires at least a cosmological constant on the brane (called brane tension) to accelerate cosmic expansion. The self-accelerating branch was later shown to be plagued by the existence of ghost excitations [10]. Without any additional modification, this leaves the normal branch as the only physically viable solution of this braneworld model, consistent with current cosmological observations of cosmic acceleration. It is this braneworld model that will be the subject of investigation in this paper.

As a model of dark energy, the normal braneworld branch exhibits an interesting generic feature of *super-acceleration* which is reflected in the phantom-like effective equation of state $w_{\text{eff}} < -1$ [11–13]. Interestingly, the Phantom brane smoothly evolves to a de Sitter stage without running into a ‘Big-Rip’ future singularity typical of conventional phantom models. Such a phantom-like equation of state appears to be consistent with the most recent set of observations of type Ia supernovae combined with other data sets [14]. The Phantom brane has a number of interesting properties: (i) it is ghost-free and is characterized by the effective equation of state $w_{\text{eff}} < -1$, (ii) for an appropriate choice of cosmological parameters,

even a spatially flat braneworld can ‘loiter’ [15], (iii) the Phantom brane possesses the remarkable property of ‘cosmic mimicry,’ wherein a high-density braneworld exhibits the *precise* expansion history of Λ CDM [16]; for reviews, see [17]. Just like the Randall–Sundrum model, this braneworld model was also used as an alternative explanation of rotation curves in galaxies without dark matter [18].

The structure of the universe on the largest scales is spectacular, and consists of a ‘cosmic web’ of intertwining galactic superclusters separated from each other by large voids. Whereas the full description of the supercluster–void complex demands a knowledge of non-linear gravitational clustering, useful insight into structure formation can already be gleaned from the linear (and weakly non-linear) approximation [19]. Linearized gravitational clustering in the braneworld model encounters obvious difficulties and complications connected with the existence of a large extra dimension. One has to take into account the corresponding dynamical degree of freedom and specify appropriate boundary conditions in the bulk space. In the simple case of a spatially flat brane, the extra dimension is noncompact, and one has to deal with its spatial infinity. The bulk gravitational effects then lead to a non-local character of the resulting equations on the brane. In spite of this difficulty, by using a very convenient Mukohyama master variable and master equation [20, 21], some progress has been made in this direction [22–29] by employing various plausible simplifying assumptions or approximations and by direct numerical integration. Most successful amongst these has been the quasi-static (QS) approximation due to Koyama and Maartens [24] which is based on the assumption of *slow temporal evolution* of (all) five-dimensional perturbations on sub-Hubble spatial scales, when compared with spatial gradients (for an extension into the non-linear regime, see [30]). The behavior of perturbations on super-Hubble spatial scales was investigated within the scaling ansatz proposed in [26] and further developed in [27, 29]. The validity of the quasi-static and scaling approximations was confirmed by numerical integration of the perturbation equations in five dimensions [28, 29].

In our previous work [31], we addressed the problem of scalar cosmological perturbations in a matter-dominated braneworld model. We considered a marginally spatially closed (with topology S^3) braneworld model, in which the ‘no-boundary’ smoothness conditions for the five-dimensional perturbations were set in the four-ball bulk space bounded by the S^3 brane. In the limit when the spatial curvature radius of the brane was large, we were able to arrive at a *closed* system of equations for scalar cosmological perturbations without any simplifying

assumptions.

Our approach differs from the semi-analytic theory and numerical computations developed in [26–29]. First of all, we only require the regularity condition in the compact bulk space but do not impose any additional boundary conditions in the bulk; in particular, we do not demand the bulk perturbations to vanish on the past Cauchy horizon of the brane. At the same time, as we have shown, due to the same regularity condition, the system of equations for perturbations becomes effectively closed on the brane and does not require integration in the bulk. This is a great simplification of the theory.

In our approach, the dynamical entities that describe perturbations on the brane are the usual matter components and the so-called Weyl fluid, or dark radiation, which stems from the projection of the five-dimensional Weyl tensor onto the brane. The closed system of dynamical equations allows one to trace the behavior of matter and Weyl-fluid perturbations once the initial conditions for these quantities are specified. In particular, for modes well inside the Hubble radius during matter domination, the matter density perturbation $\delta_m = \delta\rho_m/\rho_m$ evolves as [see also Eq. (146) below]

$$\delta_m = \mathcal{M}a + \frac{W_1}{a^{5/4}} \cos \frac{\sqrt{2}k}{aH} + \frac{W_2}{a^{5/4}} \sin \frac{\sqrt{2}k}{aH}, \quad (1)$$

where a is the scale factor, and \mathcal{M} , W_1 , W_2 are integration constants. Apart from the usual growing mode $\mathcal{M}a$, we observe two oscillating modes with decreasing amplitudes. These modes are induced by the dynamics of the Weyl fluid, or dark radiation. Note that, since these oscillatory modes have their origin in the bulk, they are absent in the scaling approximation of [26, 27, 29] or in the quasi-static approximation of [24].

Whether or not the presence of such extra modes, with dynamical origin in the bulk, can be significant for the braneworld model, depends upon the amplitudes W_1 and W_2 , and these, in turn, are determined by the primordial power spectrum of the Weyl fluid and its evolution during the radiation-dominated epoch. This calls for a development of the theory of scalar cosmological perturbations for a universe filled with several components, each with an arbitrary equation of state. Such a treatment will enable one to follow the evolution of perturbations starting from deep within the radiation-dominated regime all the way up to the current stage of accelerated expansion. This will be the main focus of the present paper.

Our paper is organized as follows. In the next section, we describe the background cosmological evolution of the normal branch of the braneworld model embedded in a flat

five-dimensional bulk. In Sec. III, we investigate the system of equations describing scalar cosmological perturbations in this model. This system is not closed on the brane because the evolution equation for the anisotropic stress from the bulk degree of freedom projected to the brane (the so-called Weyl fluid, or dark radiation) is missing. We solve this problem by proceeding to a *marginally* closed braneworld and imposing the regularity conditions in the bulk, as described in [31]. The system of equations on the brane (in the limit of a large spatial radius) now becomes closed and can therefore be used for the analysis of perturbations. In Sec. IV, we discuss possible ways of setting initial conditions for Weyl-fluid perturbations and investigate the evolution of all perturbations starting from super-Hubble scales until the end of the radiation-dominated epoch. Perturbations during matter-domination are considered in Sec. V. In Sec. VI, we present the results of numerical integration of the joint system of equations describing perturbations in radiation, pressureless matter and the Weyl fluid and compare these with Λ CDM and with the results from the quasi-static approximation. Our results are summarized in Sec. VII.

II. BACKGROUND COSMOLOGICAL EVOLUTION

Our braneworld model has the action [5, 7, 11]

$$S = M^3 \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda) - 2 \int_{\text{brane}} K \right] + \int_{\text{brane}} (m^2 R - 2\sigma) + \int_{\text{brane}} L(g_{\mu\nu}, \phi), \quad (2)$$

where \mathcal{R} is the scalar curvature of the five-dimensional bulk, and R is the scalar curvature corresponding to the induced metric $g_{\mu\nu}$ on the brane. The symbol $L(g_{\mu\nu}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric $g_{\mu\nu}$. The quantity K is the trace of the symmetric tensor of extrinsic curvature of the brane. All integrations over the bulk and brane are taken with the corresponding natural volume elements. The universal constants M and m play the role of the five-dimensional and four-dimensional Planck masses, respectively. The symbol Λ denotes the bulk cosmological constant, and σ is the brane tension.

Action (2) leads to the following effective equation on the brane [16, 32]:

$$G_{\mu\nu} + \left(\frac{\Lambda_{\text{RS}}}{b+1} \right) g_{\mu\nu} = \left(\frac{b}{b+1} \right) \frac{1}{m^2} T_{\mu\nu} + \left(\frac{1}{b+1} \right) \left[\frac{1}{M^6} Q_{\mu\nu} - \mathcal{C}_{\mu\nu} \right], \quad (3)$$

where

$$b = \frac{\sigma \ell}{3M^3}, \quad \ell = \frac{2m^2}{M^3}, \quad \Lambda_{\text{RS}} = \frac{\Lambda}{2} + \frac{\sigma^2}{3M^6} \quad (4)$$

are convenient parameters, and

$$Q_{\mu\nu} = \frac{1}{3}EE_{\mu\nu} - E_{\mu\lambda}E^\lambda{}_\nu + \frac{1}{2}\left(E_{\rho\lambda}E^{\rho\lambda} - \frac{1}{3}E^2\right)g_{\mu\nu}, \quad (5)$$

$$E_{\mu\nu} \equiv m^2 G_{\mu\nu} - T_{\mu\nu}, \quad E = E^\mu{}_\mu. \quad (6)$$

Gravitational dynamics on the brane is not closed because of the presence of the symmetric traceless tensor $\mathcal{C}_{\mu\nu}$ in (3), which stems from the projection of the five-dimensional Weyl tensor from the bulk onto the brane. We are free to interpret this tensor as the stress-energy tensor of some effective fluid, which we call the ‘Weyl fluid’ in this article (in some works, the term ‘dark radiation’ is also used).

The tensor $\mathcal{C}_{\mu\nu}$ is not freely specifiable on the brane, but is related to the tensor $Q_{\mu\nu}$ through the conservation equation

$$\nabla^\mu (Q_{\mu\nu} - M^6 \mathcal{C}_{\mu\nu}) = 0, \quad (7)$$

which is a consequence of the Bianchi identity applied to (3) and the law of stress–energy conservation for matter:

$$\nabla^\mu T_{\mu\nu} = 0. \quad (8)$$

In a universe consisting of several non-interacting components, the conservation law (8) is satisfied by each component separately.

The cosmological evolution of the Friedmann–Robertson–Walker (FRW) brane

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \quad (9)$$

can be obtained from (3) with the following result [5, 7–9, 11]:

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{2}{\ell^2} \left[1 \pm \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right]. \quad (10)$$

Here, $H \equiv \dot{a}/a$ is the Hubble parameter, $\rho = \rho(t)$ is the energy density of matter on the brane and C is a constant resulting from the presence of the symmetric traceless tensor $\mathcal{C}_{\mu\nu}$ in the field equations (3). The parameter $\kappa = 0, \pm 1$ corresponds to different spatial geometries of the maximally symmetric spatial metric γ_{ij} .

The sign ambiguity in front of the square root in equation (10) reflects the two different ways in which the bulk can be bounded by the brane [8, 11], resulting in two different branches of solutions. These are usually called the normal branch (lower sign) and the self-accelerating branch (upper sign).

In what follows, we investigate the evolution of perturbations in a marginally flat ($aH \gg 1$) normal branch of the braneworld model embedded in flat bulk spacetime (which means $C = 0$, $\Lambda = 0$). This setup will enable us to obtain a closed system of equations on the brane. The background cosmological equation (10) then reduces to

$$H^2 = \frac{\rho + \sigma}{3m^2} + \frac{2}{\ell^2} \left[1 - \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} \right)} \right] = \frac{1}{\ell^2} \left[1 - \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} \right)} \right]^2, \quad (11)$$

or, equivalently:

$$\ell H = \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} \right)} - 1. \quad (12)$$

One immediately sees that, in the regime $H \gg \ell^{-1}$, our braneworld expands like Λ CDM with the gravitational constant $8\pi G = 1/m^2$ and with the combination σ/m^2 playing the role of the cosmological constant.

Our braneworld is assumed to be filled with a multi-component fluid, with total energy density $\rho = \sum_{\lambda} \rho_{\lambda}$ and pressure $p = \sum_{\lambda} p_{\lambda}$. The usual conservation law holds for each component separately:

$$\dot{\rho}_{\lambda} + 3H\rho_{\lambda}(1 + w_{\lambda}) = 0, \quad (13)$$

where $w_{\lambda} = p_{\lambda}/\rho_{\lambda}$ is the equation of state parameter for the component labeled by λ .

In terms of the cosmological parameters

$$\Omega_{\lambda} = \frac{\rho_{\lambda 0}}{3m^2 H_0^2}, \quad \Omega_{\sigma} = \frac{\sigma}{3m^2 H_0^2}, \quad \Omega_{\ell} = \frac{1}{\ell^2 H_0^2}, \quad (14)$$

where $\ell = 2m^2/M^3$ and H_0 is the present value of the Hubble parameter, one can write the evolution equation (12) in the form

$$h(z) = \frac{H(z)}{H_0} = \sqrt{\sum_{\lambda} \Omega_{\lambda} (1+z)^{3(1+w_{\lambda})} + \Omega_{\sigma} + \Omega_{\ell}} - \sqrt{\Omega_{\ell}}, \quad (15)$$

where $1 + z = a_0/a$, and a_0 is the present value of the scale factor. The cosmological parameters are related through the equation

$$\Omega_{\sigma} = 1 + 2\sqrt{\Omega_{\ell}} - \sum_{\lambda} \Omega_{\lambda}. \quad (16)$$

For further convenience, we introduce the time-dependent parameters β and γ :

$$\beta = -2 \sqrt{1 + \ell^2 \left(\frac{\sum_{\lambda} \rho_{\lambda} + \sigma}{3m^2} \right)} = -2(1 + \ell H), \quad (17)$$

$$3\gamma - 1 = \frac{\dot{\beta}}{H\beta} = \frac{\ell \dot{H}}{H(1 + \ell H)}. \quad (18)$$

Then, from (12) and (13) one can derive a useful equation

$$\dot{H} = - \left(1 + \frac{2}{\beta} \right) \frac{\sum_{\lambda} (\rho_{\lambda} + p_{\lambda})}{2m^2}. \quad (19)$$

Restricting our attention for the moment to the present epoch, when the density of matter greatly exceeds that of radiation, we find

$$h(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_{\sigma} + \Omega_{\ell}} - \sqrt{\Omega_{\ell}}, \quad (20)$$

where

$$\Omega_{\sigma} = 1 + 2\sqrt{\Omega_{\ell}} - \Omega_m, \quad (21)$$

so that

$$\beta = -\frac{2}{\sqrt{\Omega_{\ell}}} \sqrt{\Omega_m(1+z)^3 + \Omega_{\sigma} + \Omega_{\ell}}, \quad (22)$$

$$3\gamma - 1 = -\frac{\frac{3}{2}\Omega_m(1+z)^3}{\Omega_m(1+z)^3 + \Omega_{\sigma} + \Omega_{\ell}}. \quad (23)$$

An important feature of the Phantom-brane is that, for a given value of Ω_m , its expansion rate is *slower* than that in Λ CDM, see figure 1. This property of our model is of special significance when one compares its observational predictions with observational data [33]. Indeed, recent measurements of the expansion rate at high redshifts using the data on baryon acoustic oscillations (BAO) indicate $H(z) = 222 \pm 7$ km/sec/Mpc at $z = 2.34$, which is below the value predicted by Λ CDM [34]. This tension is alleviated in the Phantom brane [33].

The *effective equation of state* (EOS) of dark energy on the brane is given by [35]

$$w(z) = \frac{\frac{2}{3}(1+z)\frac{h'}{h} - 1}{1 - \Omega_m(1+z)^3/h^2}, \quad (24)$$

where the prime denotes differentiation with respect to z . Substituting (20) into (24) one finds that the present value of the EOS is

$$w_0 \equiv w(z=0) = -1 - \frac{\Omega_m}{1 - \Omega_m} \left(\frac{\sqrt{\Omega_{\ell}}}{1 + \sqrt{\Omega_{\ell}}} \right). \quad (25)$$

Consequently, the Phantom brane possesses a phantom EOS $w_0 < -1$, just as its name suggests.

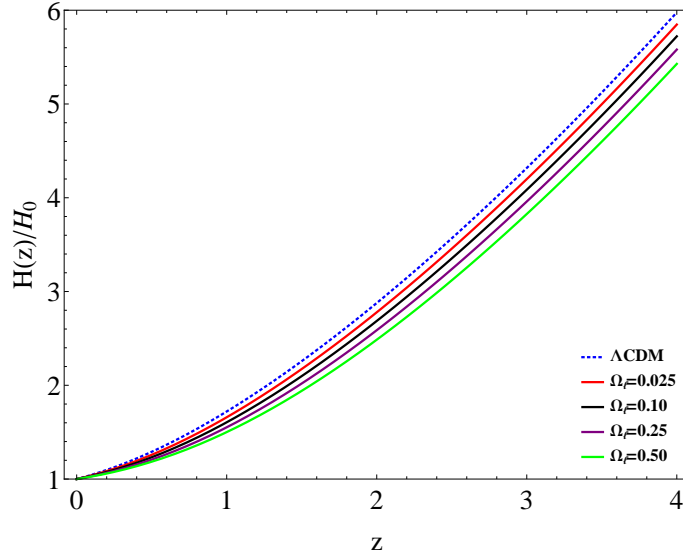


FIG. 1. The expansion rate in Λ CDM (top/dotted blue) is compared to that in the Phantom brane with $\Omega_\ell = 0.025, 0.1, 0.25, 0.5$ (top to bottom/solid), and $\Omega_m = 0.28$. One finds that past expansion is slower in the Phantom brane than in Λ CDM.

III. EVOLUTION OF SCALAR COSMOLOGICAL PERTURBATIONS

A. General equations for a multi-component fluid

Scalar cosmological perturbations of the induced metric on the brane are most conveniently described by the relativistic potentials Φ and Ψ in the longitudinal gauge:

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)\gamma_{ij}dx^i dx^j. \quad (26)$$

The components of the linearly perturbed stress-energy tensor of the λ -component of matter can be parameterized as follows:¹

$$\delta T_{(\lambda)}{}^\mu{}_\nu = \begin{pmatrix} -\delta\rho_\lambda, & -\nabla_i v_\lambda \\ \frac{\nabla^i v_\lambda}{a^2}, & \delta p_\lambda \delta^i_j + \frac{\zeta_\lambda{}^i{}_j}{a^2} \end{pmatrix}, \quad (27)$$

where $\delta\rho_\lambda$, δp_λ , v_λ , and $\zeta_{\lambda ij} = (\nabla_i \nabla_j - \frac{1}{3}\gamma_{ij} \nabla^2) \zeta_\lambda$ describe scalar perturbations.

Although the term $\rho_c = C/a^4$ [see (10)], which can be treated as the homogeneous part of the Weyl fluid, or dark radiation, was set to zero in (11), perturbations of the Weyl fluid

¹ The spatial indices i, j, \dots in purely spatially defined quantities (such as v_i and $\delta\pi_{ij}$) are always raised and lowered using the spatial metric γ_{ij} ; in particular, $\gamma^i_j = \delta^i_j$. The symbol ∇_i denotes the covariant derivative with respect to the spatial metric γ_{ij} , and the spatial Laplacian is $\nabla^2 = \nabla^i \nabla_i$.

should be taken into account. Therefore, in a similar way we introduce scalar perturbations $\delta\rho_{\mathcal{C}}$, $v_{\mathcal{C}}$, and $\delta\pi_{\mathcal{C}}$ of the traceless tensor $\mathcal{C}_{\mu\nu}$:

$$m^2\delta\mathcal{C}^\mu{}_\nu = \begin{pmatrix} -\delta\rho_{\mathcal{C}}, & -\nabla_i v_{\mathcal{C}} \\ \frac{\nabla^i v_{\mathcal{C}}}{a^2}, & \frac{\delta\rho_{\mathcal{C}}}{3}\delta^i{}_j + \frac{\delta\pi^i{}_j}{a^2} \end{pmatrix}, \quad (28)$$

where $\delta\pi_{ij} = (\nabla_i \nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2)\delta\pi_{\mathcal{C}}$.

We call v_λ and $v_{\mathcal{C}}$ the momentum potentials for the matter components and Weyl fluid, respectively; the quantities $\delta\rho_\lambda$ and $\delta\rho_{\mathcal{C}}$ are their energy-density perturbations, and ζ_λ and $\delta\pi_{\mathcal{C}}$ are the scalar potentials for their anisotropic stresses.

In this notation, the effective field equation leads to the following system for perturbations in the Fourier-space representation with respect to comoving spatial coordinates [31, 36, 37]:

$$-\frac{k^2}{a^2}\Psi = \left(1 + \frac{2}{\beta}\right) \frac{\sum_\lambda(\delta\rho_\lambda + 3Hv_\lambda)}{2m^2} + \frac{(\delta\rho_{\mathcal{C}} + 3Hv_{\mathcal{C}})}{m^2\beta}, \quad (29)$$

$$\dot{\Psi} + H\Phi = \left(1 + \frac{2}{\beta}\right) \frac{\sum_\lambda v_\lambda}{2m^2} + \frac{v_{\mathcal{C}}}{m^2\beta}, \quad (30)$$

$$\Psi - \Phi = \frac{4\delta\pi_{\mathcal{C}}}{m^2\beta(1+3\gamma)}, \quad (31)$$

where k is the comoving wavenumber and β , γ were defined in (17), (18). From the conservation law (8) applied to each component separately, we have

$$\delta\dot{\rho}_\lambda + 3H(\delta\rho_\lambda + \delta p_\lambda) = -\frac{k^2}{a^2}v_\lambda + 3(\rho_\lambda + p_\lambda)\dot{\Psi}, \quad (32)$$

$$\dot{v}_\lambda + 3Hv_\lambda = \delta p_\lambda + (\rho_\lambda + p_\lambda)\Phi. \quad (33)$$

For simplicity, we have assumed here that the anisotropic stresses of all matter components vanish: $\zeta_\lambda = 0$. Note that in general relativity these assumptions would lead to the equality of relativistic gravitational potentials Φ and Ψ , but, in the braneworld, this is not the case due to the presence of the anisotropic stress of the Weyl fluid $\delta\pi_{\mathcal{C}}$ [see (31)]. The contribution from $\delta\pi_{\mathcal{C}}$ in the braneworld model cannot be ignored, but should be established from the analysis of five-dimensional perturbations in the bulk.

The pressure perturbations δp_λ are usually decomposed into adiabatic and isentropic parts:

$$\delta p_\lambda = c_{s\lambda}^2\delta\rho_\lambda + (\rho_\lambda + p_\lambda)\Gamma_\lambda, \quad (34)$$

where

$$c_{s\lambda}^2 \equiv \frac{\dot{p}_\lambda}{\dot{\rho}_\lambda} = w_\lambda - \frac{\dot{w}_\lambda}{3H(1+w_\lambda)} \quad (35)$$

is the adiabatic sound speed, and Γ_λ describe the non-adiabatic pressure perturbations. In what follows, we restrict ourselves to adiabatic perturbations by setting $\Gamma_\lambda = 0$. At the same time, the equation of state parameters w_λ can be arbitrary functions of time.

Equation (7) serves as a conservation law for the perturbation of the Weyl fluid:

$$\delta\dot{\rho}_c + 4H\delta\rho_c = -\frac{k^2}{a^2} v_c, \quad (36)$$

$$\dot{v}_c + 3Hv_c = \frac{1}{3}\delta\rho_c + \frac{\beta(1-3\gamma)}{6} \sum_\lambda (\delta\rho_\lambda + 3Hv_\lambda) + \frac{m^2\beta k^2}{3a^2} [\Phi - 3\gamma\Psi]. \quad (37)$$

In the quasi-static approximation proposed by Koyama and Maartens in [24], there arises the following approximate relation between $\delta\pi_c$ and $\delta\rho_c$:

$$\delta\pi_c \approx \frac{a^2}{2k^2} \delta\rho_c. \quad (38)$$

Relation (38) is properly justified on sub-Hubble scales, where $k \gg aH$. As noted in Sec. I, the Koyama-Maartens relation cannot be used to study the behavior of perturbations of the Weyl fluid during the early radiative epoch on super-Hubble spatial scales.

A more general relation between $\delta\pi_c$ and $\delta\rho_c$ was derived in [31] in the limit of a marginally closed braneworld:

$$\delta\pi_c = \frac{3a^4}{2k^4} \left[\delta\ddot{\rho}_c + \left(9H - \frac{\dot{H}}{H} \right) \delta\dot{\rho}_c + \left(20H^2 + \frac{k^2}{3a^2} \right) \delta\rho_c \right]. \quad (39)$$

Relation (39) accounts for the temporal evolution of the Weyl fluid, which makes it possible to trace the evolution of perturbations right from their initial values on super-Hubble spatial scales all the way until the present time. We shall use equation (39) in the present analysis.

To investigate perturbations of a multi-component fluid, we introduce convenient variables

$$\delta_\lambda \equiv \frac{\delta\rho_\lambda}{\rho_\lambda + p_\lambda}, \quad V_\lambda \equiv \frac{v_\lambda}{\rho_\lambda + p_\lambda}. \quad (40)$$

The variables V_λ are proportional to the physical velocity potentials $V_\lambda = V_\lambda^{\text{phys}}/a$. It is reasonable to introduce similar variables for the Weyl fluid. The Weyl fluid, being described by a traceless effective stress-energy tensor $\mathcal{C}_{\mu\nu}$ in (3), behaves in a way rather similar to

radiation. Hence, since the background density of the Weyl fluid vanishes, we use the radiation background component to define

$$\delta_{\mathcal{C}} \equiv \frac{\delta\rho_{\mathcal{C}}}{\rho_r + p_r} = \frac{3\delta\rho_{\mathcal{C}}}{4\rho_r}, \quad V_{\mathcal{C}} \equiv \frac{v_{\mathcal{C}}}{\rho_r + p_r} = \frac{3v_{\mathcal{C}}}{4\rho_r}, \quad (41)$$

where ρ_r and $p_r = \rho_r/3$ are, respectively, the energy density and pressure of radiation. Naturally this assumes the presence of radiation during all stages of cosmological expansion, which is certainly true for our universe after inflation.

In terms of the new variables (40), (41), we have the following closed system of equations on the brane:

$$-\frac{k^2}{a^2} \Psi = \left(1 + \frac{2}{\beta}\right) \frac{\sum_{\lambda}(\rho_{\lambda} + p_{\lambda})(\delta_{\lambda} + 3HV_{\lambda})}{2m^2} + \frac{4\rho_r(\delta_{\mathcal{C}} + 3HV_{\mathcal{C}})}{3m^2\beta}, \quad (42)$$

$$\dot{\Psi} + H\Phi = \left(1 + \frac{2}{\beta}\right) \frac{\sum_{\lambda}(\rho_{\lambda} + p_{\lambda})V_{\lambda}}{2m^2} + \frac{4\rho_r V_{\mathcal{C}}}{3m^2\beta}, \quad (43)$$

$$\Psi - \Phi = \frac{4\delta\pi_{\mathcal{C}}}{m^2\beta(1+3\gamma)}, \quad (44)$$

$$\dot{\delta}_{\lambda} = -\frac{k^2}{a^2} V_{\lambda} + 3\dot{\Psi}, \quad (45)$$

$$\dot{V}_{\lambda} - 3Hc_{s\lambda}^2 V_{\lambda} = c_{s\lambda}^2 \delta_{\lambda} + \Phi, \quad (46)$$

$$\dot{\delta}_{\mathcal{C}} = -\frac{k^2}{a^2} V_{\mathcal{C}}, \quad (47)$$

$$\dot{V}_{\mathcal{C}} - 3\gamma HV_{\mathcal{C}} = \gamma\delta_{\mathcal{C}} + \frac{3\gamma-1}{4\rho_r} \sum_{\lambda}(\rho_{\lambda} + p_{\lambda})(\delta_{\lambda} + 3HV_{\lambda}) - \frac{k^2}{\rho_r a^2(1+3\gamma)} \delta\pi_{\mathcal{C}}, \quad (48)$$

$$\delta\pi_{\mathcal{C}} = \frac{2\rho_r a^4}{k^4} \left[\ddot{\delta}_{\mathcal{C}} + \left(1 - \frac{\dot{H}}{H^2}\right) H\dot{\delta}_{\mathcal{C}} + \frac{k^2}{3a^2} \delta_{\mathcal{C}} \right]. \quad (49)$$

Remarkably, equations (47)–(49) lead to a single equation for the variable $\delta_{\mathcal{C}}$:

$$\ddot{\delta}_{\mathcal{C}} + \left(\frac{2\beta}{2+\beta} - 3\gamma\right) H\dot{\delta}_{\mathcal{C}} + \frac{k^2}{3a^2}(2+3\gamma)\delta_{\mathcal{C}} = -\frac{k^2(1+3\gamma)}{4\rho_r a^2} \sum_{\lambda}(\rho_{\lambda} + p_{\lambda})\Delta_{\lambda}, \quad (50)$$

where

$$\Delta_{\lambda} \equiv \delta_{\lambda} + 3HV_{\lambda}. \quad (51)$$

As one can see, perturbations of all matter species influence the evolution of the Weyl fluid. In turn, perturbations of the Weyl fluid affect the gravitational potentials via (42) and (44), which influences the perturbations of matter via (45) and (46).

Using (50), (47), (48) and (49), we obtain

$$\delta\pi_{\mathcal{C}} = -\frac{2\rho_r a^2(1+3\gamma)}{3k^2} \left[\delta_{\mathcal{C}} + \frac{6HV_{\mathcal{C}}}{2+\beta} + \frac{3}{4\rho_r} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda} \right], \quad (52)$$

$$\dot{V}_{\mathcal{C}} = \left(\gamma + \frac{2}{3} \right) \delta_{\mathcal{C}} + \left(3\gamma + \frac{4}{2+\beta} \right) HV_{\mathcal{C}} + \left(\frac{3\gamma+1}{4\rho_r} \right) \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda}. \quad (53)$$

From (42) and (44), one can express the gravitational potentials in terms of the variables Δ_{λ} , V_{λ} , $\delta_{\mathcal{C}}$ and $V_{\mathcal{C}}$:

$$\Psi = -\frac{(2+\beta)a^2}{2m^2k^2\beta} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda} - \frac{4\rho_r a^2}{3m^2k^2\beta} (\delta_{\mathcal{C}} + 3HV_{\mathcal{C}}), \quad (54)$$

$$\Psi - \Phi = -\frac{8\rho_r a^2}{3m^2k^2\beta} \left[\delta_{\mathcal{C}} + \frac{6HV_{\mathcal{C}}}{2+\beta} + \frac{3}{4\rho_r} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda} \right]. \quad (55)$$

The equations for the variables Δ_{λ} and V_{λ} can be derived from (45) and (46):

$$\dot{\Delta}_{\lambda} = 3Hc_{s\lambda}^2 \Delta_{\lambda} - \frac{k^2}{a^2} V_{\lambda} + \frac{3(2+\beta)}{2m^2\beta} \sum_{\mu \neq \lambda} (\rho_{\mu} + p_{\mu})(V_{\mu} - V_{\lambda}) + \frac{4\rho_r}{m^2\beta} V_{\mathcal{C}}, \quad (56)$$

$$\dot{V}_{\lambda} = c_{s\lambda}^2 \Delta_{\lambda} + \frac{(2-\beta)a^2}{2m^2k^2\beta} \sum_{\mu} (\rho_{\mu} + p_{\mu}) \Delta_{\mu} + \frac{4\rho_r a^2}{3m^2k^2\beta} \delta_{\mathcal{C}} + \frac{4(2-\beta)\rho_r a^2}{m^2k^2\beta(2+\beta)} HV_{\mathcal{C}}. \quad (57)$$

Thus, to determine the evolution of cosmological perturbations, we need to solve the system of equations (47), (50), (56), (57). After finding solutions of this system, one can use (54), (55) to determine the gravitational potentials.

B. Perturbations of a single fluid

In this subsection, we consider the situation when one fluid component dominates over the rest. In this case, for the dominating component we can use a single-fluid version² of (56) and (57):³

$$\dot{\Delta} - 3Hc_s^2 \Delta = -\frac{k^2}{a^2} V + \frac{4\rho_r V_{\mathcal{C}}}{m^2\beta}, \quad (58)$$

$$\dot{V} = c_s^2 \Delta + \frac{(2-\beta)a^2}{2m^2k^2\beta} (\rho + p) \Delta + \frac{4\rho_r a^2}{3m^2k^2\beta} \delta_{\mathcal{C}} + \frac{4(2-\beta)\rho_r a^2}{m^2k^2\beta(2+\beta)} HV_{\mathcal{C}}. \quad (59)$$

² Some correction from sub-dominant matter component might be expected from the term $\propto (V_{\mu} - V_{\lambda})$ in (56) in the super-Hubble regime, when spatial gradients are neglected. However, for the adiabatic modes under consideration in this paper, such corrections are absent due to appropriate initial conditions; see Eq. (78) below.

³ We omit the label λ for the dominating matter component in this subsection.

Remarkably, using (58), (59) and (53), one can derive a single second-order differential equation for the variable Δ :

$$\ddot{\Delta} + (2 - 3c_s^2)H\dot{\Delta} = \left[\frac{\rho + p}{2m^2} \left(1 + \frac{6\gamma}{\beta} \right) + 3c_s^2 \left(\dot{H} + 2H^2 - \frac{k^2}{3a^2} \right) + 3H(\dot{c}_s^2) \right] \Delta + \frac{4\rho_r(1 + 3\gamma)}{3m^2\beta} \delta_c, \quad (60)$$

which should be supplemented with (50):

$$\ddot{\delta}_c + \left(\frac{2\beta}{2 + \beta} - 3\gamma \right) H\dot{\delta}_c + \frac{k^2}{3a^2}(2 + 3\gamma)\delta_c = -\frac{k^2(1 + 3\gamma)}{4\rho_r a^2}(\rho + p)\Delta. \quad (61)$$

The system of equations (60), (61) describes perturbations of a relativistic fluid with arbitrary equation of state $w = p/\rho$ and whose adiabatic sound speed c_s^2 is given by (35).

Once the solutions for Δ and δ_c are known, one can find the momentum potentials for matter and Weyl fluid, namely V, V_c , via the relations (47) and (58). After that, the gravitational potentials Φ and Ψ can be determined from (54) and (55):

$$\Psi = -\frac{(2 + \beta)a^2}{2m^2 k^2 \beta}(\rho + p)\Delta - \frac{4\rho_r a^2}{3m^2 k^2 \beta}(\delta_c + 3HV_c), \quad (62)$$

$$\Psi - \Phi = -\frac{8\rho_r a^2}{3m^2 k^2 \beta} \left[\delta_c + \frac{6HV_c}{2 + \beta} + \frac{3}{4\rho_r}(\rho + p)\Delta \right]. \quad (63)$$

Finally, the evolution of all other (sub-dominant) fluid components is described by (45) and (46) with the gravitational potentials (62), (63) as source terms.

C. The Friedmann expansion regime

An important feature of our braneworld, which distinguishes it from the Randall–Sundrum model, is that the effect of the extra dimension on cosmic expansion is usually small at early times [7, 11]. We refer to this early epoch as the *Friedmann regime*, since the equations of (3 + 1)-dimensional general relativity determine the course of cosmic expansion during this early time [see Eq. (11)]. Nevertheless, at the perturbative level, perturbations of the extra-dimensional Weyl fluid exist at all times and can never be ignored. Thus, we investigate the evolution of perturbations during early times when

$$\sum_{\lambda} \rho_{\lambda} + \sigma \gg \frac{m^2}{\ell^2}, \quad H \gg \ell^{-1}, \quad (64)$$

implying that the effect of the extra dimension (parameterized by the inverse length $\ell^{-1} = M^3/2m^2$) on background evolution is small. In this approximation, equation (11) turns into the Friedmann expansion law

$$H^2 \approx \frac{1}{3m^2} \left(\sum_{\lambda} \rho_{\lambda} + \sigma \right), \quad (65)$$

so that

$$\dot{H} \approx -\frac{\sum_{\lambda}(\rho_{\lambda} + p_{\lambda})}{2m^2}, \quad \beta \approx -2\ell H, \quad \gamma \approx \frac{1}{3} \left(1 + \frac{\dot{H}}{H^2} \right) \approx -\frac{\sum_{\lambda} \rho_{\lambda}(1 + 3w_{\lambda}) - 2\sigma}{6(\sum_{\lambda} \rho_{\lambda} + \sigma)}. \quad (66)$$

Relation (61), which describes the evolution of the Weyl-fluid, simplifies to

$$\ddot{\delta}_C + (2 - 3\gamma) H \dot{\delta}_C + \frac{k^2}{3a^2} (2 + 3\gamma) \delta_C = -\frac{k^2(1 + 3\gamma)}{4\rho_r a^2} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda}, \quad (67)$$

where we have neglected terms of order $1/\beta$ with respect to unity, according to (64), (66). Perturbations of the energy density in each fluid component under this approximation can be derived from (56), (57):

$$\dot{\Delta}_{\lambda} = 3H c_{s\lambda}^2 \Delta_{\lambda} - \frac{k^2}{a^2} V_{\lambda} + \frac{3}{2m^2} \sum_{\mu \neq \lambda} (\rho_{\mu} + p_{\mu}) (V_{\mu} - V_{\lambda}) + \frac{4\rho_r}{m^2 \beta} V_C, \quad (68)$$

$$\dot{V}_{\lambda} = c_{s\lambda}^2 \Delta_{\lambda} - \frac{a^2}{2m^2 k^2} \sum_{\mu} (\rho_{\mu} + p_{\mu}) \Delta_{\mu} + \frac{4\rho_r a^2}{3m^2 k^2 \beta} (\delta_C - 3H V_C), \quad (69)$$

where the variable V_C is related to δ_C via (47).

Evolution of the gravitational potentials during the Friedmann regime is determined by [see (54) and (55)]:

$$-\frac{k^2}{a^2} \Psi = \frac{1}{2m^2} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda} + \frac{4\rho_r}{3m^2 \beta} (\delta_C + 3H V_C), \quad (70)$$

$$\Psi - \Phi = -\frac{2a^2}{m^2 k^2 \beta} \sum_{\lambda} (\rho_{\lambda} + p_{\lambda}) \Delta_{\lambda} - \frac{8\rho_r a^2}{3m^2 k^2 \beta} \left(\delta_C + \frac{6H V_C}{\beta} \right). \quad (71)$$

Finally, we note that in the case of a *single-component* fluid, we can, instead of (68) and (69), employ the early-time version of (60):

$$\ddot{\Delta} + (2 - 3c_s^2) H \dot{\Delta} + \left[\dot{H} - 3c_s^2 \left(\dot{H} + 2H^2 - \frac{k^2}{3a^2} \right) - 3H(\dot{c}_s^2) \right] \Delta = \frac{4\rho_r(1 + 3\gamma)}{3m^2 \beta} \delta_C. \quad (72)$$

One finds that, in the formal limit of $|\beta| \rightarrow \infty$, perturbations of the Weyl fluid do not affect those of ordinary matter. Thus, we expect that perturbations of matter components

at early times will behave as in general relativity. However, the approximation $|\beta| \rightarrow \infty$ is too crude and does not allow control of its accuracy [in contrast to the approximation $|\beta| \approx 2\ell H \gg 1$, which was used to derive (67)–(72)].

Below, in Sec. IV, we analyze the evolution of cosmological perturbations during the epoch of radiation domination. We shall discover that the approximation (72) is quite accurate during such early times.

D. Scaling approximation on super-Hubble scales

We can observe that, at the early stages of cosmological evolution, when the Friedmann approximation considered in the previous subsection is applicable, our approach matches well with the scaling ansatz considered in [26, 27, 29]. Indeed, in the Friedmann regime, the expansion of the brane is driven by the energy density of the dominating matter component [see (65)] which evolves by a power law in the scale factor a . One can expect the existence of solution of (67) for the variable δ_C in the form of a power of a as well. In such a case, we have the order-of-magnitude estimates

$$\dot{\delta}_C \sim H\delta_C, \quad \ddot{\delta}_C \sim H^2\delta_C. \quad (73)$$

On super-Hubble scales, where $k^2 \ll a^2 H^2$, the last term on the left-hand side of (67) can be neglected. If we also neglect the homogeneous part of solution of (67) (this condition is equivalent to that there is no sources for δ_C except the brane itself), we obtain

$$\delta_C \sim \frac{k^2(1+3\gamma)\rho}{a^2 H^2 \rho_r} \Delta, \quad (74)$$

where ρ and Δ are both related to the dominating matter component. To be more specific, in the era of matter domination, we have $H^2 \propto \rho_m \propto a^{-3}$, $(1+3\gamma) \approx 1/2$, and $\Delta_m \propto a$, which results in $\delta_C \propto a^3$. In the case of radiation domination, we have $H^2 \propto \rho_r \propto a^{-4}$, $(1+3\gamma) \propto \rho_m/\rho_r \propto a$ [see also (90)], and $\Delta_r \propto a^2$. Correspondingly, $\delta_C \propto a^5$ in this case.⁴

Taking into account (47), we also have the order-of-magnitude estimate

$$V_C = -\frac{a^2}{k^2} \dot{\delta}_C \sim \frac{a^2 H}{k^2} \delta_C. \quad (75)$$

⁴ We note that the variable δ_C is related to the master variable (projected onto the brane) Θ_b via $\Theta_b = -\frac{4a^5 \rho_r}{m^2 k^4} \delta_C$ (see [31]). As follows from our consideration, the master variable on the brane behaves as $\Theta_b \propto a^p$, where $p = 6$ in the regime of radiation domination, and $p = 4$ if pressureless matter dominates over radiation. We observe that the powers p in the evolution law of the variable Θ_b coincide with those predicted by the scaling ansatz in [26].

Then, we can apply (70) and (71) to establish the following relation between the gravitational potentials:

$$\frac{\Psi - \Phi}{\Psi} = -\frac{2}{\ell H} \left[1 + \mathcal{O}\left(\frac{1}{\ell H}\right) + \mathcal{O}\left(\frac{k^2}{a^2 H^2}\right) \right]. \quad (76)$$

Thus, in the regime of Friedmann expansion, which is characterized by the condition $\ell H \gg 1$, perturbations on super-Hubble spatial scales are described by the approximate relation

$$\frac{\Psi - \Phi}{\Psi} \approx -\frac{2}{\ell H}. \quad (77)$$

Relation (77) can be considered as a super-horizon counterpart of (39), because, if we *assume* it, we get a closed system of equations for perturbations on the brane. Remarkably, the scaling ansatz for braneworld perturbations gives the same closing relation on super-Hubble scales [29], which indicates the match between the two methods, at least to the leading order in a small parameter $1/\ell H \ll 1$. In this sense, the scaling ansatz, which is based on the assumption of vanishing bulk master variable on the past Cauchy horizon, can be regarded as an approximate partial solution of a more general condition (39). Our condition (39) allows one to investigate the behavior of perturbations which have been originated purely in the bulk, along with perturbations originated purely on the brane. A rigorous definition of these two modes will be given in the next section.

IV. PERTURBATIONS DURING THE RADIATIVE EPOCH

A. Initial conditions

The primordial spectra for scalar cosmological perturbations are specified deep within the radiation domination epoch. At that time, the modes relevant to structure formation belong to super-Hubble spatial scales, and perturbations of pressureless matter⁵ are decoupled from those of radiation.⁶ As is well known, in general relativity, adiabatic non-decaying modes⁷ on super-Hubble scales remain almost constant in time, and are related to the value of the gravitational potential as follows:⁸

$$\delta_{m(i)} = \delta_{r(i)} = -\frac{3\Phi_{(i)}}{2}, \quad V_{m(i)} = V_{r(i)} = \frac{\Phi_{(i)}}{2H}, \quad \Psi_{(i)} = \Phi_{(i)} = \text{const}, \quad (78)$$

⁵ Pressureless matter in our investigation possesses all characteristics of cold dark matter.

⁶ Ultra-relativistic primordial plasma will be treated as an ideal radiation. Effects related to baryons and neutrino will not be considered in this work.

⁷ In this paper, we do not investigate possible effects of isocurvature modes.

⁸ In terms of energy density contrasts this relation implies: $\frac{\delta\rho_{m(i)}}{\rho_{m(i)}} = \frac{3}{4} \left(\frac{\delta\rho_{r(i)}}{\rho_{r(i)}} \right) = -\frac{3\Phi_{(i)}}{2}$.

where the subscript ‘ (i) ’ denotes initial values, the subscript ‘ m ’ refers to pressureless matter, and the subscript ‘ r ’ to radiation.

As the effects of the extra dimension in our model weaken at early times, we expect the above relations to also be valid in our braneworld during the radiative epoch (this expectation will be confirmed in the next section). We also assume that initial linear perturbations of matter and radiation are random with Gaussian statistics. In this case, they are completely characterized by the power spectrum $\mathcal{P}_m(k)$, defined as

$$\langle \delta_{m(i)}(\mathbf{k}) \delta_{m(i)}(\mathbf{k}') \rangle = \langle \delta_{r(i)}(\mathbf{k}) \delta_{r(i)}(\mathbf{k}') \rangle = \frac{\mathcal{P}_m(k)}{4\pi k^3} \delta(\mathbf{k} + \mathbf{k}'). \quad (79)$$

Cosmological observations, interpreted within the framework of the Λ CDM model, indicate that the initial power spectrum is nearly flat. The following parametrization is commonly used:

$$\mathcal{P}_m(k) = A_m \left(\frac{k}{k_*} \right)^{n_s - 1}, \quad (80)$$

where k_* is a pivot scale which, in the Planck data analysis [38], is chosen to be $k_*/a_0 = 0.05 \text{ Mpc}^{-1}$, A_m defines the normalization of the spectrum, and $n_s - 1$ is its slope. The power spectrum is flat if the scalar spectral index $n_s = 1$, which is very close to the observed value $n_s \approx 0.96$ [38].

We will see in the next section that perturbations $\delta_{\mathcal{C}}$ of the Weyl fluid, as well as those of matter, weakly depend on time before Hubble-radius crossing. It is thus natural to assume that the initial value of the Weyl fluid $\delta_{\mathcal{C}(i)}$ is also randomly distributed with Gaussian statistics, so that

$$\langle \delta_{\mathcal{C}(i)}(\mathbf{k}) \delta_{\mathcal{C}(i)}(\mathbf{k}') \rangle = \frac{\mathcal{P}_{\mathcal{C}}(k)}{4\pi k^3} \delta(\mathbf{k} + \mathbf{k}'), \quad (81)$$

where the primordial power spectrum $\mathcal{P}_{\mathcal{C}}(k)$ for the Weyl fluid can, in principle, be different from $\mathcal{P}_m(k)$ for matter (and radiation). Setting aside the issue of generation of primordial perturbations, one can study the consequences of a general parametrization of the preceding type (80):

$$\mathcal{P}_{\mathcal{C}}(k) = A_{\mathcal{C}} \left(\frac{k}{k_*} \right)^{\alpha}, \quad (82)$$

where k_* is the same pivot scale as in (80), and $A_{\mathcal{C}}$ is the normalization of the initial power spectrum for the Weyl fluid.

In this paper we do not discuss possible mechanisms for the generation of primordial perturbations in our braneworld, and therefore cannot tell whether or not the primordial

perturbations of the Weyl fluid are correlated with those of matter and radiation. For simplicity we shall assume them to be statistically independent. In this case, one can consider the evolution of two basic modes:

Brane mode: initial perturbations of the Weyl fluid are absent:

$$A_m = A_0, \quad A_C = 0; \quad (83)$$

Bulk mode: initial perturbations in matter & radiation on the brane are absent:

$$A_m = 0, \quad A_C = cA_0. \quad (84)$$

Here, $A_0 = 2.2 \times 10^{-9}$ defines the normalization of the primordial power spectrum, and c is a numerical constant describing the intensity of the bulk mode. In the case of superposition of these two modes, the power spectrum of any quantity will be given by the sum of the corresponding power spectra (since these two modes are assumed to be statistically independent).

We note that, if the initial adiabatic perturbation for matter/radiation and for the Weyl fluid are not assumed to be statistically independent, say, if $\delta_{C(i)}$ is proportional to $\delta_{r(i)}$ due to a common mechanism for their primordial generation (which we do not discuss in this work), then the power spectra would be calculated by squaring a superposition of the two modes.

In the following section, we consider in more detail the evolution of perturbations on super-Hubble spatial scales in our braneworld model.

B. Evolution of perturbations prior to Hubble crossing

At early times, cosmological evolution is dominated by an ultra-relativistic component (which has equation of state $w_r = c_{s(r)}^2 = 1/3$ and can be treated as radiation). The background cosmological equation (15) in this case can be approximated by the relations:

$$H \approx H_0 \sqrt{\Omega_r} (1+z)^2, \quad \dot{H} \approx -2H^2, \quad \gamma \approx -\frac{1}{3}, \quad (85)$$

where Ω_r is the cosmological parameter corresponding to radiation. Radiation dominates as long as $\rho_r \gg \rho_m$, where ρ_m is the pressureless matter density characterized by the

cosmological parameter Ω_m . This condition is valid as long as⁹

$$\frac{a}{a_0} \ll \frac{\Omega_r}{\Omega_m} \simeq 2.5 \times 10^{-4}. \quad (86)$$

Obviously, the influence of the cosmological parameter Ω_σ in (15) can also be ignored at this stage of cosmological evolution.

The leading braneworld corrections to (85) are of the order of magnitude $1/\ell H$, where the quantity ℓH is estimated as

$$\ell H \approx \sqrt{\frac{\Omega_r}{\Omega_\ell}} \left(\frac{a_0}{a}\right)^2 \gg \frac{\Omega_m^2}{\sqrt{\Omega_\ell \Omega_r^3}} \simeq 8.5 \times 10^5. \quad (87)$$

Since $1/\ell H \ll 1$, the universe expands as in general relativity and perturbations during the radiative regime can be analyzed using the results of Sec. III C.

In this case, from (72) and (67) we obtain a system of two equations:

$$\ddot{\Delta}_r + H\dot{\Delta}_r - \left(2H^2 - \frac{k^2}{3a^2}\right) \Delta_r = -\frac{2H^2(1+3\gamma)}{\ell H} \delta_c, \quad (88)$$

$$\ddot{\delta}_c + 3H\dot{\delta}_c + \frac{k^2}{3a^2} \delta_c = -\frac{k^2(1+3\gamma)}{3a^2} \Delta_r. \quad (89)$$

We are going to show that the right-hand sides of these equations can be ignored, by comparing them to some terms on the left-hand sides of the corresponding equation. To do this, we need to estimate the factor $(1+3\gamma)$ more precisely. Taking into account that the next correction to (85) comes from the pressureless matter component, we compute

$$1+3\gamma \approx \frac{\Omega_m}{2\Omega_r} \frac{a}{a_0} \ll 1. \quad (90)$$

Next, we shall assume that, deep in the radiative regime, the following condition is satisfied:

$$(1+3\gamma) \ll \left| \frac{\delta_c}{\Delta_r} \right| \ll \frac{\ell H}{(1+3\gamma)}. \quad (91)$$

Under this condition, the system of equations (88), (89) simplifies to

$$\ddot{\Delta}_r + H\dot{\Delta}_r - \left(2H^2 - \frac{k^2}{3a^2}\right) \Delta_r \approx 0, \quad (92)$$

$$\ddot{\delta}_c + 3H\dot{\delta}_c + \frac{k^2}{3a^2} \delta_c \approx 0. \quad (93)$$

⁹ Here and below, for numerical estimates we assume the following values of the cosmological parameters [33]: $\Omega_m = 0.28$, $\Omega_r = 7 \times 10^{-5}$, and $\Omega_\ell = 0.025$. In this case, $\Omega_\sigma \approx 1.036$.

Perturbations with any given wavenumber k begin their evolution deep inside the super-Hubble regime, where $k \ll aH$. Equations (92) and (93) can easily be solved in the formal limit $k \rightarrow 0$:

$$\Delta_r = \Delta_{r(i)} \left(\frac{a}{a_0} \right)^2 + \Delta_{r(d)} \frac{a_0}{a}, \quad \delta_{\mathcal{C}} = \delta_{\mathcal{C}(i)} + \delta_{\mathcal{C}(d)} \frac{a_0}{a}, \quad (94)$$

where $\Delta_{r(i)}$, $\Delta_{r(d)}$, $\delta_{\mathcal{C}(i)}$ and $\delta_{\mathcal{C}(d)}$ are constants of integration.

The terms proportional to a_0/a in (94) correspond to decaying modes. The influence of decaying modes on future evolution is assumed to be negligibly small, so, in what follows, these modes shall be neglected.

Let us perform more careful analysis of (92) and (93), allowing for non-zero k . Introduce a new variable:

$$x \equiv \frac{k}{\sqrt{3} a H} = \frac{s}{\sqrt{3 \Omega_r}} \left(\frac{a}{a_0} \right)^2, \quad s \equiv \frac{k}{a_0 H_0} = \frac{2\pi}{\lambda_0 H_0}. \quad (95)$$

Here, λ_0 is the spatial scale of perturbation at the present time. From the viewpoint of structure formation, the most relevant values of s are

$$10^2 \lesssim s \lesssim 10^6, \quad (96)$$

where the lower value corresponds to supercluster scales ($\lambda_0 \sim 100$ Mpc), and the upper value corresponds to galactic scales ($\lambda_0 \sim 10$ kpc).

In terms of the variable x , equations (92) and (93) are written as

$$x^2 \Delta_r'' - (2 - x^2) \Delta_r \approx 0, \quad (97)$$

$$x^2 \delta_{\mathcal{C}}'' + 2\delta_{\mathcal{C}}' + x^2 \delta_{\mathcal{C}} \approx 0, \quad (98)$$

where the prime denotes differentiation with respect to x . In the region $x \ll 1$ (which corresponds to $k \ll aH$), we can look for solution of (97), (98) in the form of an asymptotic expansion:

$$\Delta_r \approx \delta_{r(i)} x^2 \left[1 - \frac{x^2}{10} + \mathcal{O}(x^4) \right], \quad (99)$$

$$\delta_{\mathcal{C}} \approx \delta_{\mathcal{C}(i)} \left[1 - \frac{x^2}{6} + \mathcal{O}(x^4) \right]. \quad (100)$$

Other variables can be calculated to their leading order as

$$\Delta_r \approx \delta_{r(i)} \left(1 - \frac{k^2}{30a^2H^2} \right) \frac{k^2}{3a^2H^2}, \quad \delta_r \approx -3HV_r \approx \delta_{r(i)} \left[1 + \mathcal{O} \left(\frac{k^2}{a^2H^2} \right) \right], \quad (101)$$

$$\delta_{\mathcal{C}} \approx \delta_{\mathcal{C}(i)} \left(1 - \frac{k^2}{18a^2 H^2} \right) \approx \delta_{\mathcal{C}(i)}, \quad V_{\mathcal{C}} \approx \frac{\delta_{\mathcal{C}(i)}}{9H} \left[1 + \mathcal{O} \left(\frac{k^2}{a^2 H^2} \right) \right], \quad (102)$$

where we have used (47) and (58) and assumed that

$$\left| \frac{\delta_{\mathcal{C}}}{\Delta_r} \right| \ll \frac{3\ell H}{2}. \quad (103)$$

Condition (103) can be written in the form

$$\left| \frac{\delta_{\mathcal{C}(i)}}{\delta_{r(i)}} \right| \ll \frac{s^2}{2\sqrt{\Omega_\ell \Omega_r}} \simeq (3.8 \times 10^2) s^2, \quad (104)$$

which seems to be quite a realistic restriction on $\delta_{\mathcal{C}(i)}$ for values of s in the range (96). Violation of (104), in fact, threatens a breakdown of the linear approximation. The validity of (104) also ensures condition (91) at this stage of cosmological evolution. Thus, the approximate solutions (101) and (102) are justified.

As we see, the variables δ_r and $\delta_{\mathcal{C}}$ are both constant in time for modes which lie outside the Hubble radius during the radiation dominated epoch. Using (70) and (71), we can now estimate braneworld effects on the evolution of the gravitational potentials. We start with the potential Ψ :

$$\Psi = -\frac{2\delta_{r(i)}}{3} \left(1 - \frac{4\sqrt{\Omega_r \Omega_\ell}}{s^2} \frac{\delta_{\mathcal{C}(i)}}{\delta_{r(i)}} \right) \approx -\frac{2\delta_{r(i)}}{3}, \quad (105)$$

where the approximation is valid due to (104).

To evaluate the potential Φ , we compute

$$\frac{\Psi - \Phi}{\Psi} \approx -\frac{2}{\ell H} - \frac{6\sqrt{\Omega_r \Omega_\ell}}{s^2} \frac{\delta_{\mathcal{C}(i)}}{\delta_{r(i)}}. \quad (106)$$

From (87) and (104), we find $|(\Psi - \Phi)/\Psi| \ll 1$ prior to Hubble-radius crossing. Thus neither the perturbation in radiation nor the gravitational potentials are significantly affected by the Weyl fluid during this period of cosmological evolution. As a result, our braneworld has the same initial relations between the perturbation of radiation and gravitational potentials as those in general relativity:

$$\Psi_{(i)} \approx \Phi_{(i)} \approx -\frac{2\delta_{r(i)}}{3}. \quad (107)$$

Using (101), we can improve this result to include corrections in k/aH :

$$\Psi \approx \Phi \approx -\frac{2\delta_{r(i)}}{3} \left(1 - \frac{k^2}{30a^2 H^2} \right). \quad (108)$$

In passing, we note that relation (106) coincides with the scaling ansatz prediction (77) if the bulk mode is neglected ($\delta_{\mathcal{C}(i)} = 0$).

The evolution of perturbations in pressureless matter, whose background density is subdominant in the radiative regime, can be determined from the conservation laws (45), (46), in which the gravitational potentials Ψ and Φ act as sources:

$$\dot{\delta}_m = -\frac{k^2}{a^2} V_m + 3\dot{\Psi}, \quad (109)$$

$$\dot{V}_m = \Phi. \quad (110)$$

The system of equations (109), (110) can easily be solved with Ψ and Φ given by (108):

$$\delta_m \approx \delta_{m(i)} - \frac{7k^2}{20a^2H^2} \Phi_{(i)}, \quad V_m \approx \frac{\Phi_{(i)}}{2H} \left[1 - \frac{k^2}{60a^2H^2} \right]. \quad (111)$$

For the adiabatic mode, the initial values $\delta_{m(i)}$ and $\delta_{r(i)}$ are equal [see (78)]. Thus, in the leading approximation, we have

$$\delta_m \approx \delta_{m(i)} = -\frac{3\Phi_{(i)}}{2}, \quad V_m \approx \frac{\Phi_{(i)}}{2H}, \quad \Delta_m \equiv \delta_m + 3HV_m \approx -\frac{3k^2}{8a^2H^2} \Phi_{(i)}. \quad (112)$$

The initial power spectra for the statistically independent quantities $\Phi_{(i)}$ and $\delta_{c(i)}$ are specified in Sec. IV A.

C. Evolution of perturbations after Hubble crossing

After Hubble crossing, when $k > aH$, the behavior of perturbations significantly changes. Modes most relevant for structure formation [see (96)] all cross the horizon during the radiation-dominated epoch. In this section, we consider the evolution of perturbations well after Hubble crossing.

Perturbations during the radiative regime are described by the general system of equations (88) and (89). First, we obtain the homogeneous solution of these equations (i.e., without their right-hand sides):

$$\Delta_r^{(\text{hom})} = 3\delta_{r(i)} \left[\frac{\sqrt{3}aH}{k} \sin\left(\frac{k}{\sqrt{3}aH}\right) - \cos\left(\frac{k}{\sqrt{3}aH}\right) \right], \quad (113)$$

$$\delta_c^{(\text{hom})} = \delta_{c(i)} \frac{\sqrt{3}aH}{k} \sin\left(\frac{k}{\sqrt{3}aH}\right), \quad (114)$$

where (113) and (114) have been matched with the non-decaying solutions (101) and (102) at the moment of Hubble-radius crossing, in order to fix the constants of integration (keeping in mind that decaying modes have been neglected). Expression (113) coincides with the general-relativistic solution in this regime.

Let us determine the evolution of modes deep inside the Hubble radius, where $k \gg aH$. Specifically, we shall restrict ourselves to the time period when

$$4 \times 10^3 \ll \frac{a_0}{a} \ll (1.2 \times 10^2)s, \quad (115)$$

when radiation is still dominating ($\rho_r \gg \rho_m$) and $k \gg aH$. The homogeneous solutions (113) and (114) now take the form

$$\Delta_r^{(\text{hom})} = -3 \delta_{r(i)} \cos \left(\frac{k}{\sqrt{3}aH} \right), \quad (116)$$

$$\delta_{\mathcal{C}}^{(\text{hom})} = \delta_{\mathcal{C}(i)} \frac{\sqrt{3}aH}{k} \sin \left(\frac{k}{\sqrt{3}aH} \right). \quad (117)$$

Let us discuss the domain of validity of these solutions. We note that the right-hand side of (88) can be neglected under the condition¹⁰

$$\frac{\delta_{\mathcal{C}}^{(\text{hom})}}{\Delta_r^{(\text{hom})}} \ll \frac{k^2}{6a^2H^2} \left(\frac{\ell H}{1+3\gamma} \right) \Rightarrow \frac{\delta_{\mathcal{C}(i)}}{\delta_{r(i)}} \ll \frac{s^3}{\Omega_m \sqrt{3}\Omega_\ell} \simeq 13s^3. \quad (118)$$

Violation of (118), again, would threaten the validity of linear approximation. Thus, the evolution of Δ_r on sub-Hubble scales is reasonably described by (116).

The right-hand side of (89) can be neglected under the condition

$$\frac{\delta_{\mathcal{C}}^{(\text{hom})}}{\Delta_r^{(\text{hom})}} \gg (1+3\gamma) \Rightarrow \left(\frac{a_0}{a} \right)^2 \gg \frac{\sqrt{3}\Omega_m s}{2\Omega_r^{3/2}} \cdot \frac{\delta_{r(i)}}{\delta_{\mathcal{C}(i)}} \simeq (4 \times 10^5) s \frac{\delta_{r(i)}}{\delta_{\mathcal{C}(i)}}. \quad (119)$$

We note that this estimate would always be satisfied on sub-Hubble scales if, initially,

$$\left| \frac{\delta_{\mathcal{C}(i)}}{\delta_{r(i)}} \right| \gtrsim 2.5 \times 10^{-2} s. \quad (120)$$

However, estimate (120) can be violated, which means that the impact of Δ_r on the evolution of $\delta_{\mathcal{C}}$ cannot be neglected. We can account for this influence by solving (89) with Δ_r determined in (116):

$$\ddot{\delta}_{\mathcal{C}} + 3H\dot{\delta}_{\mathcal{C}} + \frac{k^2}{3a^2} \delta_{\mathcal{C}} = \frac{\delta_{r(i)} k^2 (1+3\gamma)}{a^2} \cos \left(\frac{k}{\sqrt{3}aH} \right). \quad (121)$$

Introducing a new variable $x \equiv k/(\sqrt{3}aH)$ and using relations (85) and (90), we can transform (121) to

$$x^2 \delta_{\mathcal{C}}'' + 2x \delta_{\mathcal{C}}' + x^2 \delta_{\mathcal{C}} = B x^3 \cos x, \quad B \equiv \frac{3\sqrt{3}\Omega_m \delta_{r(i)}}{2\sqrt{\Omega_r} s}, \quad (122)$$

¹⁰ Here and below, we perform estimates for the ratio of amplitudes of quantities oscillating around zero.

where the prime denotes differentiation with respect to x . One can easily find the general solution of this equation in the limit $x \gg 1$ (corresponding to $k \gg aH$):

$$\delta_{\mathcal{C}} \approx \frac{\delta_{\mathcal{C}(i)}}{x} \sin x + \frac{Bx^2}{6} \sin x = \delta_{\mathcal{C}}^{(\text{bulk})} + \delta_{\mathcal{C}}^{(\text{brane})}, \quad (123)$$

where we have introduced the notation

$$\delta_{\mathcal{C}}^{(\text{brane})} = \delta_{r(i)} \left(\frac{k(1+3\gamma)}{2\sqrt{3}aH} \right) \sin \left(\frac{k}{\sqrt{3}aH} \right), \quad (124)$$

$$\delta_{\mathcal{C}}^{(\text{bulk})} = \delta_{\mathcal{C}(i)} \left(\frac{\sqrt{3}aH}{k} \right) \sin \left(\frac{k}{\sqrt{3}aH} \right). \quad (125)$$

Here, the function $\delta_{\mathcal{C}}^{(\text{brane})}$ represents the behavior of $\delta_{\mathcal{C}}$ in the brane mode, while $\delta_{\mathcal{C}}^{(\text{bulk})}$ is its behavior in the bulk mode¹¹ [see the definition in Sec. IV A; equations (83) and (84)]. We observe that the amplitude of the brane mode grows with time, in contrast to the behavior of the bulk mode, which has decreasing amplitude in this regime.

As established earlier, the contribution from the bulk mode to the evolution of Δ_r is negligibly small, which means that the bulk mode of Δ_r can be neglected:

$$\Delta_r \approx \Delta_r^{(\text{brane})} = -3 \delta_{r(i)} \cos \left(\frac{k}{\sqrt{3}aH} \right). \quad (126)$$

The brane mode of $\delta_{\mathcal{C}}$ (124) also might affect Δ_r via (88). This effect, however, is negligible because of the condition

$$\frac{\delta_{\mathcal{C}}^{(\text{brane})}}{\Delta_r^{(\text{brane})}} \ll \frac{k^2}{6a^2H^2} \left(\frac{\ell H}{1+3\gamma} \right) \Rightarrow \frac{\sqrt{3}k}{aH} \frac{\ell H}{(1+3\gamma)^2} \gg 1, \quad (127)$$

which is obviously valid in the regime under consideration, characterized by $k \gg aH$, $\ell H \gg 1$ and $(1+3\gamma) \ll 1$. Consequently, the evolution of Δ_r is described by the brane mode (126) with high accuracy.

Finally, we are in a position to investigate the influence of the Weyl-fluid perturbation on the gravitational potentials. From (70), we have

$$-\frac{k^2}{a^2} \Psi \approx \frac{2\rho_r \Delta_r}{3m^2} \left(1 - \frac{\delta_{\mathcal{C}}}{\ell H \Delta_r} \right), \quad (128)$$

¹¹ We should note that expression (125) represents the dominating, but not full, contribution from the bulk mode. Even if $\delta_{r(i)} = 0$, still we have some contribution to $\delta_{\mathcal{C}}$ from Δ_r , which appears due to the back-reaction from $\delta_{\mathcal{C}}$ in (88).

where we have used the estimate $H|V_C| \ll |\delta_C|$, which follows from the definition (47), solution (123) and condition $k \gg aH$. In the regime under consideration, we can also make estimates

$$\frac{1}{\ell H} \frac{\delta_C^{(\text{brane})}}{\Delta_r} = \frac{\Omega_m \sqrt{\Omega_\ell} s}{12\sqrt{3} \Omega_r^2} \left(\frac{a}{a_0}\right)^4 \ll 1.7 \times 10^{-9} s \quad (129)$$

for the brane mode (124), and

$$\frac{1}{\ell H} \frac{\delta_C^{(\text{bulk})}}{\Delta_r} = \sqrt{\frac{\Omega_\ell}{3}} \left(\frac{a}{a_0}\right) s^{-1} \left| \frac{\delta_{C(i)}}{\delta_{r(i)}} \right| \ll 2.3 \times 10^{-5} s^{-1} \left| \frac{\delta_{C(i)}}{\delta_{r(i)}} \right| \quad (130)$$

for the bulk mode (125).

Consequently, the correction to the general-relativistic result in (128) is negligibly small if

$$\left| \frac{\delta_{C(i)}}{\delta_{r(i)}} \right| \lesssim 4.3 \times 10^4 s \quad \text{and} \quad s \lesssim 5.9 \times 10^8. \quad (131)$$

Using (71), we can also verify that

$$\left| \frac{\Psi - \Phi}{\Psi} \right| \ll 1 \quad (132)$$

under the same conditions (131).

For reasonable initial conditions that do not invalidate the linear approximation [compare with (104)], and for values of s relevant to structure formation [see (96)], both conditions in (131) are satisfied, and we can conclude that the Weyl fluid does not significantly affect the gravitational potentials during the radiation-dominated epoch.

Perturbations of pressureless matter during radiation domination are described by the system of equations (109), (110), in which the gravitational potentials Φ and Ψ act as sources. Since the gravitational potentials are unaffected by the Weyl fluid, the same is true for pressureless matter perturbations, which thus evolve according to the general-relativistic law [39]:

$$\delta_m \approx -9\Phi_{(i)} \left(\log \frac{k}{\sqrt{3}aH} + \mathbf{C} - \frac{1}{2} \right), \quad (133)$$

where $\mathbf{C} = 0.577 \dots$ is Euler's constant.

Summarizing this subsection, we have established the behavior of Weyl-fluid perturbations in the sub-Hubble regime of the radiation-dominated epoch [relation (123)]. Since the amplitude of the bulk mode in (123) decreases with time, while the amplitude of the brane mode grows, one expects the brane mode to dominate during future epochs. Matter perturbations in this regime are dominated by the brane mode, which exhibits general-relativistic

behavior. Deviations from general-relativistic behavior can, however, be caused by the brane mode during the future matter-dominated epoch, which we discuss in the next section.

V. PERTURBATIONS DURING MATTER-DOMINATION

After transition from radiation domination to matter domination, we can once more consider perturbations in a single-component fluid. Thus, to describe the evolution of cosmological perturbations during this epoch, we use equations (60) and (61) with $c_s^2 = w = 0$:

$$\ddot{\Delta}_m + 2H\dot{\Delta}_m - \frac{\rho_m}{2m^2} \left(1 + \frac{6\gamma}{\beta}\right) \Delta_m = \frac{4\rho_r(1+3\gamma)}{3m^2\beta} \delta_c, \quad (134)$$

$$\ddot{\delta}_c + \left(\frac{2\beta}{2+\beta} - 3\gamma\right) H\dot{\delta}_c + \frac{k^2}{3a^2}(2+3\gamma)\delta_c = -\frac{k^2(1+3\gamma)\rho_m}{4a^2\rho_r} \Delta_m, \quad (135)$$

where the time-dependent parameters β and γ are defined in (17) and (18), respectively.

The system of equations (134), (135) (in slightly different notation) was thoroughly investigated in [31]. It was argued there that the problem greatly simplifies during the Friedmann regime of expansion when braneworld effects do not contribute significantly to cosmic expansion and can be ignored. This will be discussed in the next subsection.

A. Matter perturbations during the Friedmann regime

Background cosmological evolution during the matter-dominated Friedmann regime is described by the approximate relations of Sec. III C:

$$H \approx H_0 \sqrt{\Omega_m} (1+z)^{3/2}, \quad \dot{H} \approx -\frac{3}{2} H^2, \quad \gamma \approx -\frac{1}{6}, \quad (136)$$

where Ω_m is the cosmological parameter corresponding to pressureless matter. One should note that the effect of the cosmological constant has been neglected, which is valid while

$$(1+z)^3 \gg \frac{\Omega_\sigma}{\Omega_m} \approx 3.7. \quad (137)$$

Using (136), we can estimate:

$$\ell H \approx \sqrt{\frac{\Omega_m}{\Omega_\ell}} (1+z)^{3/2} \approx 3.3 (1+z)^{3/2}. \quad (138)$$

Obviously, the validity of (137) justifies the early-time Friedmann regime [see condition (64)] and, therefore, relations (136).

The early-time version of (134), (135), which is valid when $z \gg 1$, reads:

$$\ddot{\Delta}_m + 2H\dot{\Delta}_m - \frac{\rho_m}{2m^2}\Delta_m = -\frac{\rho_r}{3m^2\ell H}\delta_{\mathcal{C}}, \quad (139)$$

$$\ddot{\delta}_{\mathcal{C}} + \frac{5H}{2}\dot{\delta}_{\mathcal{C}} + \frac{k^2}{2a^2}\delta_{\mathcal{C}} = -\frac{k^2\rho_m}{8a^2\rho_r}\Delta_m. \quad (140)$$

The system of equations (139), (140) is valid on all spatial scales. We consider small and large spatial scales separately.

1. Matter perturbations on super-Hubble spatial scales

The behavior of perturbations on super-Hubble spatial scales in matter-dominated regime is important for the large-scale modes that remain outside the horizon during the whole radiation-dominated epoch.

For the zeroth order solution, by setting $k = 0$ and neglecting the decaying mode, we have the following approximate solution of (140):

$$\delta_{\mathcal{C}} \approx \delta_{\mathcal{C}(i)}. \quad (141)$$

The constant $\delta_{\mathcal{C}(i)}$ in this approximation is the same as in (94).

Taking into account (136) and (141), we can present the general solution of (139) in the form

$$\Delta_m \approx \mathcal{M}_0 a + \frac{\rho_r}{\ell H \rho_m} \delta_{\mathcal{C}(i)}, \quad (142)$$

where the decaying mode is again neglected. The first term here ($\propto a$) represents the usual general-relativistic result, and \mathcal{M}_0 is a constant of integration related to the initial value of matter perturbation $\delta_{r(i)}$ from (101). The second term (which evolves as \sqrt{a}) is a correction from the bulk mode (the definitions of the brane and bulk modes were given in Sec. IV A).

At the second step of iteration, we can substitute (142) into the right-hand side of (140) to find a correction to the zero-order result (141). In doing so, we are mostly interested in the correction coming from the component $\Delta_m = \mathcal{M}_0 a$, because it gives a leading contribution from the brane mode to the perturbation of the Weyl fluid. As a result, we obtain

$$\delta_{\mathcal{C}} \approx \delta_{\mathcal{C}(i)} - \frac{k^2\rho_m}{96a^2H^2\rho_r}\mathcal{M}_0 a. \quad (143)$$

We note that the brane mode of the Weyl fluid (corresponding to $\delta_{\mathcal{C}(i)} = 0$) behaves in accordance with the scaling ansatz considered in [26, 27, 29] and discussed in Sec. III D:

$\delta_{\mathcal{C}} \propto a^3$. Solutions (142) and (143) can be used to evaluate the gravitational potentials via (70) and (71):

$$\frac{\Psi - \Phi}{\Psi} \approx -\frac{2}{\ell H}, \quad (144)$$

which confirms our previous estimate (77).

We observe that deviations of the evolution of brane perturbations from the general-relativistic behavior is insignificant on super-Hubble spatial scales (due to the relation $\ell H \gg 1$ valid in the Friedmann regime under consideration) *provided only the brane mode is taken into account*. The possible presence of the bulk mode requires special attention. In the following section, we will study the effect of the bulk mode for perturbations already in the sub-Hubble regime at the moment of transition from radiation domination to matter domination.

2. Matter perturbations on sub-Hubble spatial scales

It was shown in [31] that, deep in the matter-dominated regime, and for sufficiently large amplitudes of $\delta_{\mathcal{C}}$, the right-hand side of (140) can be neglected both on super-Hubble and sub-Hubble spatial scales,¹² and only the homogeneous part of the general solution for the Weyl-fluid perturbations in this regime can significantly influence the dynamics of matter perturbations. On sub-Hubble spatial scales, the solution can be presented in the form

$$\delta_{\mathcal{C}} \approx \left(\frac{aH}{k}\right)^{3/2} \left(F \cos \frac{\sqrt{2}k}{aH} + G \sin \frac{\sqrt{2}k}{aH} \right), \quad (145)$$

where the constants of integration F and G can be calculated via matching (145) with (125) for the bulk mode and with (124) for the brane mode at the matter–radiation equality.

Using (145), we can solve (139) and find the correction from the bulk mode to the general-relativistic result $\Delta_m = \mathcal{M}a$ on sub-Hubble spatial scales:

$$\Delta_m \approx \mathcal{M}a + \frac{2\Omega_r\sqrt{\Omega_\ell}}{\Omega_m s} \left(\frac{aH}{k}\right)^{5/2} \left(F \cos \frac{\sqrt{2}k}{aH} + G \sin \frac{\sqrt{2}k}{aH} \right), \quad (146)$$

where s was defined in (95).

We note here that both the brane mode and bulk mode give contribution to the constants \mathcal{M} , F , and G in (146). (See (83), (84) for a description of brane and bulk modes.) However,

¹² This is not true at later times of cosmological evolution, when the quasi-static approximation is applicable.

For the quasi-static approximation, see Sec. V B.

as will be revealed in the following analysis, the contribution of the bulk mode to the constant \mathcal{M} is much smaller than that of the brane mode for all reasonable initial conditions. The Weyl-fluid perturbation (145) decreases as $a^{-3/4}$, and the oscillatory perturbation of pressureless matter [the second term in (146)] as $a^{-5/4}$. The brane mode $\Delta_m^{(\text{brane})} = \mathcal{M}a$ in (146) is thus dominating in the full solution for pressureless matter perturbations at early times. Thus, significant deviation from the early-time regime, (145), can be expected only at late times, when the condition $H \gg \ell^{-1}$ ceases to be valid and bulk effects come into play. Equations (134) and (135) at this period of evolution can be integrated only numerically.

More accurate results for perturbation growth can be obtained by numerically integrating the exact system of equations, including perturbations of matter, radiation and the Weyl fluid, starting from the time well before Hubble-radius crossing. This procedure will be implemented in Sec. VI.

B. The Quasi-static approximation

In the quasi-static approximation, time derivatives of perturbations in the Weyl fluid are assumed to be much smaller than spatial gradients (on sub-Hubble scales) and are neglected [this reduces (39) to (38)]. Under this condition, Eq. (135) reduces to

$$\delta_c \approx - \frac{3\rho_m(1+3\gamma)}{4\rho_r(2+3\gamma)} \Delta_m, \quad (147)$$

which results in a closed equation for Δ_m in (134), namely

$$\ddot{\Delta}_m + 2H\dot{\Delta}_m \approx \frac{\rho_m\Delta_m}{2m^2} \left(1 + \frac{1}{3\mu}\right). \quad (148)$$

Here, we have introduced a new parameter μ to put this equation in the form in which it is presented in [24]:

$$\mu \equiv 1 + \ell H \left(1 + \frac{\dot{H}}{3H^2}\right) = - \frac{\beta(2+3\gamma)}{6}. \quad (149)$$

The evolution of the gravitational potentials in the quasi-static approximation can be determined from (62) and (63):

$$- \frac{k^2}{a^2} \Psi \approx \frac{\rho_m\Delta_m}{2m^2} \left(1 - \frac{1}{3\mu}\right), \quad (150)$$

$$- \frac{k^2}{a^2} \Phi \approx \frac{\rho_m\Delta_m}{2m^2} \left(1 + \frac{1}{3\mu}\right). \quad (151)$$

Note that, in the quasi-static approximation, one obtains a closed second-order differential equation (148) for matter perturbation which does not depend on the spatial scale explicitly [see (134) and (135) for comparison], although the initial conditions can be scale-dependent. From (150) and (151) it is evident that, under the quasi-static approximation, $(\Psi - \Phi)/\Psi$ and Φ/Ψ are also scale independent,

$$\frac{\Psi - \Phi}{\Psi} = -\frac{2}{3\mu - 1}, \quad \frac{\Phi}{\Psi} = \frac{3\mu + 1}{3\mu - 1}. \quad (152)$$

The quasi-static approximation implies that only the brane mode is significant for matter perturbations at late times, while the contribution from the bulk mode appears to be negligibly small. This result is consistent with the considerations of Secs. IV C and V A, where we have seen that the bulk mode rapidly decays after the Hubble-radius crossing. Numerical integration of the exact system of equations also confirms the convergence to the quasi-static approximation, as will be demonstrated in Sec. VI. This is also in agreement with the results of [26–29], which confirm the quasi-static regime on small spatial scales.

VI. RESULTS OF NUMERICAL INTEGRATION

In Sec. III A, we derived a closed system of equation for perturbations of a multi-component fluid on the brane. These equations can be numerically integrated, and one can obtain the joint evolution of perturbations of pressureless matter, radiation and the Weyl fluid.¹³

Equations describing perturbations of matter and radiation densities can be derived from (56) and (57):

$$\dot{\Delta}_m = -\frac{k^2}{a^2} V_m + \frac{2(2+\beta)\rho_r}{m^2\beta} (V_r - V_m) + \frac{4\rho_r}{m^2\beta} V_c, \quad (153)$$

$$\dot{V}_m = \frac{(2-\beta)a^2}{2m^2k^2\beta} \left(\rho_m \Delta_m + \frac{4\rho_r}{3} \Delta_r \right) + \frac{4\rho_r a^2}{3m^2k^2\beta} \delta_c + \frac{4(2-\beta)\rho_r a^2}{m^2k^2\beta(2+\beta)} H V_c, \quad (154)$$

$$\dot{\Delta}_r = H \Delta_r - \frac{k^2}{a^2} V_r + \frac{3(2+\beta)\rho_m}{2m^2\beta} (V_m - V_r) + \frac{4\rho_r}{m^2\beta} V_c, \quad (155)$$

$$\dot{V}_r = \frac{\Delta_r}{3} + \frac{(2-\beta)a^2}{2m^2k^2\beta} \left(\rho_m \Delta_m + \frac{4\rho_r}{3} \Delta_r \right) + \frac{4\rho_r a^2}{3m^2k^2\beta} \delta_c + \frac{4(2-\beta)\rho_r a^2}{m^2k^2\beta(2+\beta)} H V_c, \quad (156)$$

¹³ In this paper, we treat the pressureless matter as a single component, neglecting all effects specific to baryonic matter.

while the variables δ_C and V_C evolve according to (50) and (47):

$$\ddot{\delta}_C + \left(\frac{2\beta}{2+\beta} - 3\gamma \right) H \dot{\delta}_C + \frac{k^2}{3a^2} (2+3\gamma) \delta_C = - \frac{k^2(1+3\gamma)}{4\rho_r a^2} \left(\rho_m \Delta_m + \frac{4\rho_r}{3} \Delta_r \right), \quad (157)$$

$$- \frac{k^2}{a^2} V_C = \dot{\delta}_C. \quad (158)$$

The parameters H , β and γ are determined by the background cosmological equations (15)–(18), in which the pressureless matter density ρ_m and the radiation density ρ_r are expressed in terms of Ω_m and Ω_r , respectively.

The system of equations (153)–(158) can be numerically integrated from an early epoch when the relevant mode was in the super-Hubble regime $k \ll aH$. The initial behavior of mode functions during this period was derived in Sec. IV B [see (101), (102), (107) and (112)], and is specified by two initial amplitudes $\delta_{m(i)} = \delta_{r(i)}$, $\delta_{C(i)}$ with power spectra (80) and (82), respectively.

After the integration of equations (153)–(158), one determines the gravitational potentials by using the general relations (54) and (55), which become, respectively:

$$\Psi = - \frac{(2+\beta)a^2}{2m^2 k^2 \beta} \left(\rho_m \Delta_m + \frac{4\rho_r}{3} \Delta_r \right) - \frac{4\rho_r a^2}{3m^2 k^2 \beta} \left(\delta_C + 3HV_C \right), \quad (159)$$

$$\Psi - \Phi = - \frac{8\rho_r a^2}{3m^2 k^2 \beta} \left(\Delta_r + \frac{3\rho_m}{4\rho_r} \Delta_m + \delta_C + \frac{6HV_C}{2+\beta} \right). \quad (160)$$

In figure 2, we present the results of our numerical integration for late-time perturbations of pressureless matter for the brane mode and for the bulk modes with two amplitudes of the power spectrum. For simplicity, we choose the scale-invariant initial power spectrum for the Weyl fluid perturbation, i.e., we set $\alpha = 0$ in (82), while numerically calculating the bulk mode. One can see that, even for an extremely high initial perturbations of δ_C , the late-time perturbations of matter are dominated by the brane mode. As long as the first condition in (131) is satisfied, the bulk-mode contribution will be insignificant compared to that of the brane mode. [See (83) and (84) for a description of brane and bulk modes.]

Concentrating then on the brane mode, we find that late-time perturbations of matter follow quite well the quasi-static approximation due to Koyama and Maartens, described in Sec. V B by equation (148). This behavior is illustrated in figure 3 for the wave-number $s = 2\pi \cdot 400 \simeq 2500$, which corresponds to the spatial scale of 10 Mpc. This figure shows: (i) the growth of perturbations on the brane can differ significantly from those in Λ CDM. (ii) Perturbations on the brane with wavelength of 10 Mpc match the quasi-static approximation

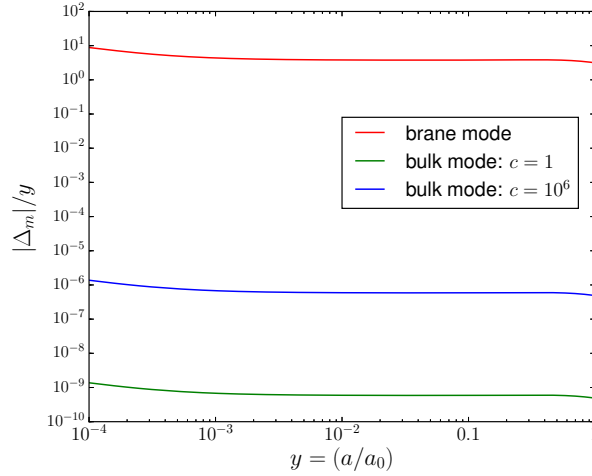


FIG. 2. Density perturbations on the brane are shown for three cases: (i) Topmost (red) line shows the evolution of matter density perturbations when perturbations in the Weyl fluid are initially absent (the brane mode). (ii) The middle (blue) line shows matter density perturbations when initial perturbations in matter are absent but perturbations in the Weyl fluid are present (the bulk mode). (iii) The lower (green) curve is the same as (ii) but with a suppressed initial amplitude for perturbations in the Weyl fluid. In all three cases, $\Delta_m \propto a(t)$ during the matter-dominated epoch. Here, the constant c is defined in (84). One can see that, even for extremely large initial perturbations of the Weyl fluid (with $c = 10^6$), the brane mode dominates during the late-time evolution. For numerical illustration, we have chosen $\Omega_\ell = 0.025$ and $s = 2\pi \cdot 400$ which corresponds to ~ 10 Mpc scale. [See (83) and (84) for a description of brane and bulk modes.]

quite well. As one increases Ω_ℓ , perturbations on the brane show greater departure from those in Λ CDM. This is illustrated in figure 4 for the single spatial scale of 10 Mpc. This dependency on Ω_ℓ provides a potent observational test for the braneworld. (Note that in the limit $\Omega_\ell \rightarrow 0$, the braneworld reduces to Λ CDM). The difference between the braneworld and Λ CDM is shown in figure 5 for various scales. One finds that the ratio $\Delta_m/\Delta_m^{\Lambda\text{CDM}}$ increases very slowly for smaller scales (higher values of s) and saturates for $s \gtrsim 100$, which includes the range of scales of interest, given in (96). This indicates the self-similarity of Δ_m for these scales. Note that under the quasi-static approximation, the ratio $\Delta_m/\Delta_m^{\Lambda\text{CDM}}$ does not depend upon spatial scale, as Eq. (148) differs from the corresponding equation of Λ CDM only by a scale-independent factor $[1 + 1/(3\mu)]$. Hence, in view of figure 5, one can conclude that the quasi-static approximation is most accurate on smaller spatial scales

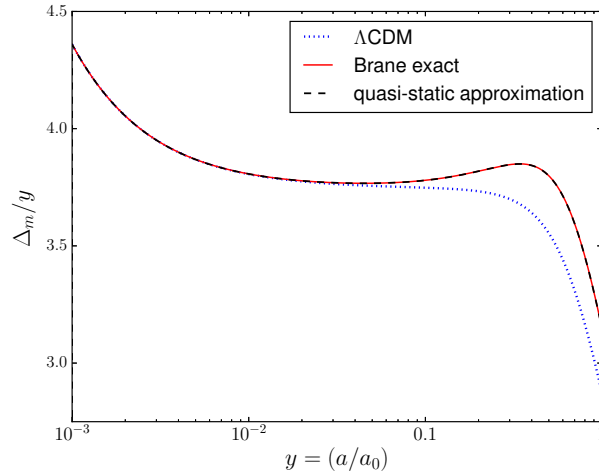


FIG. 3. Perturbations on the brane obtained using the exact system of equations (solid red) are compared with those in Λ CDM (dotted blue) and the quasi-static approximation of Koyama–Maartens (dashed black). The brane parameter is $\Omega_\ell = 0.025$ and $s \equiv k/a_0 H_0 = 2\pi \cdot 400 \simeq 2500$ is the comoving wavelength. Also see figures 5 and 6, in which the scale dependence of brane perturbations is highlighted.

while being less accurate on very large spatial scales ($s < 100$). This conclusion is explicitly evident from figure 6, where the accuracy of the quasi-static approximation is shown¹⁴ as a function of the dimensionless wavenumber $s \equiv k/(a_0 H_0)$ for various values of Ω_ℓ . This is the manifestation of the fact that, on smaller spatial scales (higher k or s), the derivatives of δ_C are more strongly suppressed in (135).

From (125) and (145) it is clear that the Weyl fluid perturbation, δ_C , in the bulk mode, will oscillate with decaying amplitude inside the horizon, and that this will remain true for any initial $\delta_{C(i)}$. During the matter domination epoch, δ_C (in both brane and bulk modes) grows due to the back reaction from Δ_m in (135), but this effect is expected to be significant only at late times when Δ_m has grown sufficiently. However the growth in δ_C is slow enough to satisfy the quasi-static approximation (147), as shown in figure 7 for the brane mode. Note that the small oscillations in δ_C , shown in figure 7, are triggered by the back reaction from sub-horizon scale oscillating perturbations in radiation, Δ_r , during the radiative epoch, see (124).

¹⁴ In figure 6, we calculate Δ_m and Δ_m^{qs} with the same initial conditions, $\Delta_m = a/a_0$ and $\dot{\Delta}_m = \dot{a}/a_0$, while starting from $a/a_0 = 0.001$. Furthermore, we calculate Δ_m using the system of equations (134) and (135) valid during matter domination.

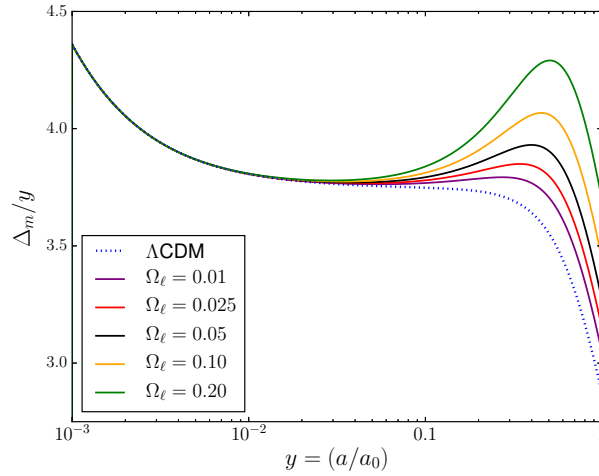


FIG. 4. Density perturbations on the brane are very sensitive to the value of the brane parameter Ω_ℓ , which depends on the ratio of the five- and four-dimensional gravitational couplings, see (4) and (14). At late times, perturbations on the brane grow more rapidly than in Λ CDM (dotted blue line). The above results correspond to the scale $s \equiv k/a_0 H_0 \simeq 2500$. Unlike perturbations in Λ CDM, perturbations on the brane are very weakly scale dependent. This is illustrated in figure 5, in which Ω_ℓ is held fixed while s is allowed to vary, and in figure 6, in which both s and Ω_ℓ are varied.

The corresponding evolution of the gravitational potential Ψ is shown in figure 8(a), and that of the relative difference $(\Psi - \Phi)/\Psi$ in figure 8(b). Figure 9(a) shows the evolution of $(\Psi - \Phi)/\Psi$ for different values of Ω_ℓ , whereas figure 9(b) shows the present value of $(\Psi - \Phi)/\Psi$ as a function of Ω_ℓ for different values of the wavenumber $s \equiv k/a_0 H_0$. The corresponding results for the ratio Φ/Ψ are shown in figure 10. These figures clearly demonstrate the following features of our model:

1. The potentials Φ and Ψ depart from the Λ CDM behaviour, $\Phi = \Psi$, at late times.
2. The departure from Λ CDM is more pronounced for larger values of the parameter Ω_ℓ , which is defined by (4) and (14) and which depends on the ratio of the five- and four-dimensional gravitational couplings.
3. The dependence of Φ and Ψ on scale is very weak, as shown in figures 9(b) and 10(b). For large values of $s \gtrsim 100$, the results saturate and coincide with those of the quasi-static approximation. Hence, quasi-static approximation is able to reproduce the exact

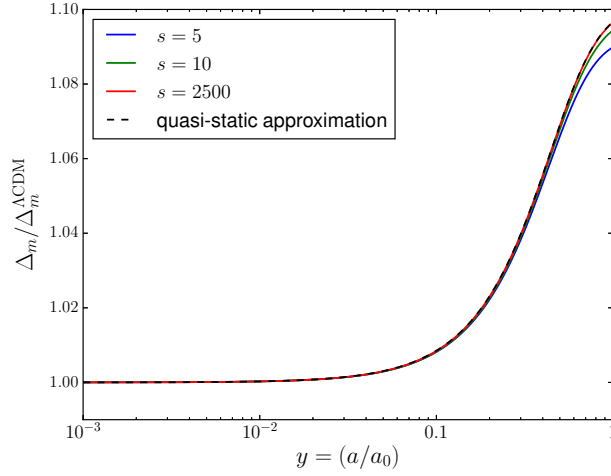


FIG. 5. Density perturbations on the brane are shown relative to those in Λ CDM. Perturbations on the brane are weakly scale-dependent. The parameter $s = k/a_0 H_0$ is the ratio of the Hubble length scale to the comoving length scale. We set $\Omega_\ell = 0.025$ for all values of s . Note that the evolution on larger spatial scales, with $s \gtrsim 100$, saturates and agrees with the quasi-static approximation extremely well.

results for larger values of the wavenumber $s \equiv k/a_0 H_0 \gtrsim 100$, including the scales relevant for structure formation, given in (96). For very large spatial scales, $s < 100$, the quasi-static approximation becomes less accurate, consistent with figures 5 and 6.

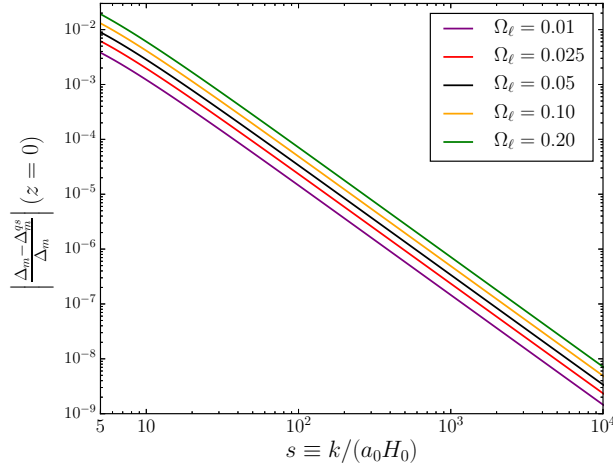


FIG. 6. The fractional difference between perturbations on the brane obtained by solving the exact system of equations (Δ_m) and by using the quasi-static approximation (Δ_m^{qs}) is shown at the present epoch ($z = 0$) for different values of $\Omega_\ell = 0.025$ and $s = k/a_0 H_0$. One notes that the accuracy of the quasi-static approximation increases for higher values of s and lower values of Ω_ℓ . (The limit $\Omega_\ell \rightarrow 0$ corresponds to Λ CDM.)

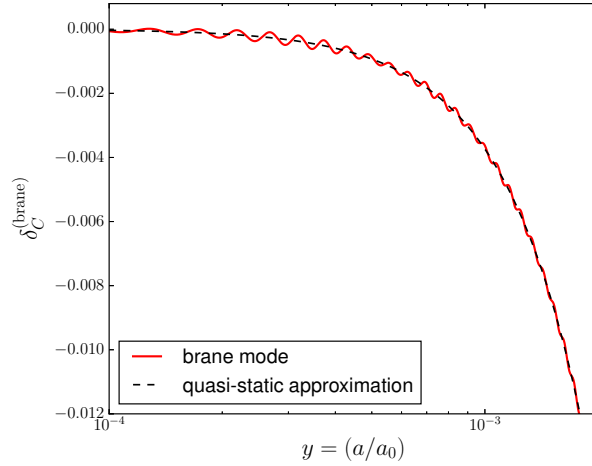


FIG. 7. Perturbations of the Weyl fluid (in the brane mode) for $\Omega_\ell = 0.025$ and $s \equiv k/a_0 H_0 \simeq 2500$. The slow growth of the Weyl-fluid perturbations ensures the validity of the quasi-static approximation. Small oscillations at early times are caused by the back-reaction from perturbations in radiation, described by the Δ_r term in (124). Note that they are not captured by the quasi-static approximation.

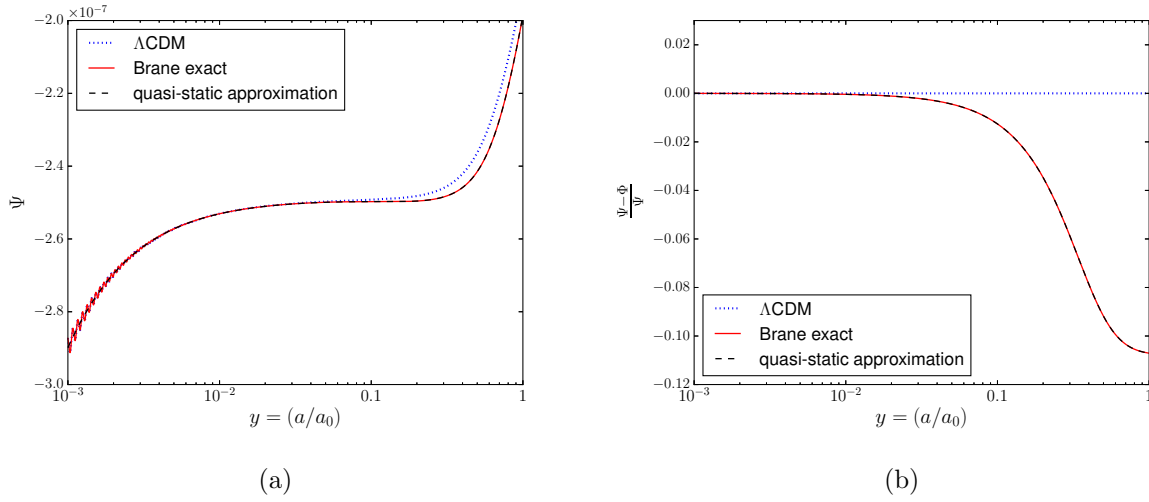


FIG. 8. **(a)**: Evolution of the gravitational potential Ψ . Oscillations at early times are due to perturbations in radiation and are sourced by the Δ_r term in (159). Such oscillations are also present in Λ CDM. **(b)**: Evolution of the relative difference between the gravitational potentials Ψ and Φ . The fractional difference $(\Psi - \Phi)/\Psi$ is very sensitive to the value of Ω_ℓ and marginally sensitive to the value of s , as illustrated in figures 9(a) & 9(b). Note the excellent accuracy of the quasi-static approximation. Here we used $\Omega_\ell = 0.025$ and $s \equiv k/a_0 H_0 \simeq 2500$ for the numerical illustrations.

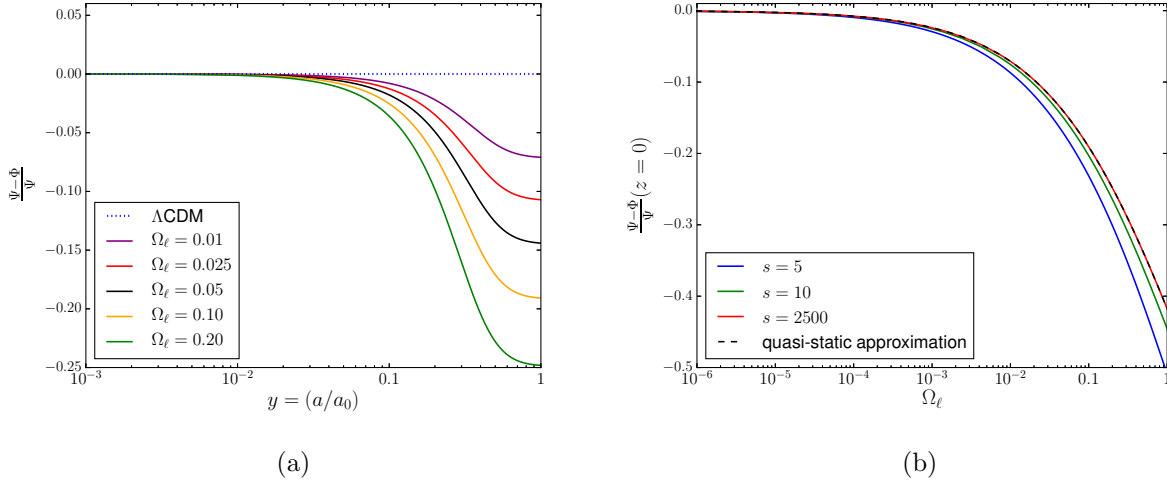


FIG. 9. **(a):** The evolution of the relative difference between the gravitational potentials is shown for various values of the brane parameter Ω_ℓ . Note that the difference $\Psi - \Phi$ increases with the increase in Ω_ℓ , where Ω_ℓ , defined by (4) and (14), depends on the ratio of the five- and four-dimensional gravitational couplings. Our results are shown for the length scale $s \equiv k/a_0 H_0 \simeq 2500$. **(b):** The relative difference between the gravitational potentials, evaluated at the present epoch, is shown as a function of the brane parameter Ω_ℓ for different values of the wavenumber s . For very large spatial scales, $s < 100$, the fractional difference between potentials deviates from that of the quasi-static approximation, given in (152). On the other hand, for $s \gtrsim 100$, the result saturates and converges to the quasi-static approximation.

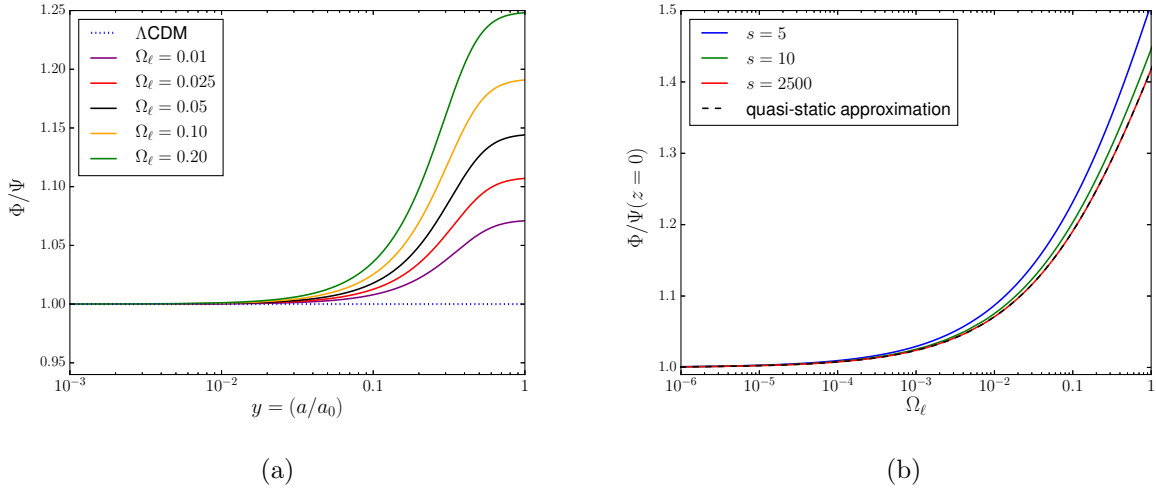


FIG. 10. **(a)**: The evolution of the ratio Φ/Ψ is shown for different values of the brane parameter Ω_ℓ . Note that Φ/Ψ increases with Ω_ℓ , where Ω_ℓ , defined in (4) and (14), depends on the ratio of the five- and four-dimensional gravitational couplings. By contrast, $\Phi = \Psi$ in Λ CDM (dotted blue line). Results are shown for the length scale $s \equiv k/a_0 H_0 \simeq 2500$. **(b)**: The current value of Φ/Ψ is shown as a function of Ω_ℓ for different values of the length scale $s \equiv k/a_0 H_0$. Note that the quasi-static approximation provides an excellent fit to the full analysis for $s \gtrsim 100$.

VII. CONCLUSIONS

We have investigated the evolution of perturbations on the normal branch of the induced gravity braneworld, in which the brane is embedded in a flat bulk space-time. Of special interest to us was the behavior of the bulk mode of perturbations, which is characterized by non-zero (and possibly quite large) initial amplitude of effective Weyl fluid perturbations.

Our approach to the problem, which is described in Sec. III, goes beyond the quasi-static approximation and allows one to study the behavior of perturbations starting from deep within the radiation-dominated epoch, where the corresponding modes are super-Hubble. In Sec. IV B, we established that perturbations of the Weyl fluid and those of matter perturbations depend very weakly on time in the super-Hubble regime. This allows one to set initial conditions during this early cosmological epoch, as described in Sec. IV A. We consider the initial perturbations of matter and of the Weyl fluid as being statistically independent. In this case, the initial conditions describe two natural modes, which we refer to as the ‘brane mode’ and the ‘bulk mode’, depending upon which perturbation vanishes initially. The behavior of the *brane mode* on super-Hubble spatial scales is shown to be in good agreement with the predictions of the scaling approximation [26, 27, 29]. On sub-Hubble spatial scales, both methods converge to the regime well described by the quasi-static approximation of [24]. The *bulk mode* originates from perturbations with nonzero initial conditions in the bulk; it was usually ignored in other approaches.

In Sec. IV C, we established that Weyl-fluid perturbations decrease with time in the bulk mode while growing in the brane mode after the Hubble-radius crossing during radiation domination. At the same time, perturbations of radiation, as well as those of the gravitational potentials Φ and Ψ , demonstrate the general-relativistic behavior, $\Phi \simeq \Psi$, in this regime.

At the beginning of the matter-dominated epoch, the contribution from the bulk mode continues to decrease, as argued in Sec. V A. This explains the validity of the quasi-static approximation, which was described in Sec. V B.

All our results are confirmed by numerical integration of the exact system of equations for pressureless matter, radiation and the Weyl fluid, which is performed in Sec. VI. They are also in good agreement with the five-dimensional numerical simulations of [28, 29].

Our main conclusion is that the presence of oscillatory terms in the evolution of pertur-

bations (like those described by equation (146)) are not likely to be of any significance, and the main effects of the braneworld ansatz would be the almost self-similar (Ω_ℓ -dependent) smooth deviation from general-relativistic behavior, shown in figures 4, 8(a), 8(b), 9(a).

Our main results are summarized below.

1. Perturbations on the brane grow more rapidly than in the Λ CDM model at late times. This was illustrated in figure 4. Departure from Λ CDM is more pronounced for larger values of the parameter Ω_ℓ , which is defined in (4) and (14) and which depends on the ratio of the five- and four-dimensional gravitational couplings. (Note that the braneworld model under consideration passes to Λ CDM in the limit $\Omega_\ell \rightarrow 0$.)
2. Departure from Λ CDM is also reflected in the behaviour of the potentials Φ and Ψ which follow the Λ CDM asymptote, $\Phi = \Psi$, only at early times. At late times, corresponding to $z \lesssim 50$, the difference between Φ and Ψ becomes pronounced, and this effect is larger for larger values of Ω_ℓ .
3. The evolution of density perturbations and of the potentials Φ and Ψ displays a very weak dependence on length scale. This was illustrated in figures 5 and 6 for the density contrast Δ_m , and in figures 9(b) and 10(b) for the ratios $(\Psi - \Phi)/\Psi$ and Φ/Ψ , respectively. These figures illustrate that our results for Δ_m , Φ and Ψ converge to those of the quasi-static approximation for larger values of the wavenumber $s \equiv k/a_0 H_0 \gtrsim 100$, which encompass the scales relevant for the large-scale structure formation, given in (96).

The results of this work will be compared with observations in a companion paper.

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